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IZA DP No. 6248

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December 2011

Forschungsinstitut
zur Zukunft der Arbeit
Institute for the Study
of Labor

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Discussion Paper No. 6248
December 2011

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IZA Discussion Paper No. 6248
December 2011

ABSTRACT

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In order to formalize the act of agreement between two individuals, the concept of *consensus functional equation*, for a bi-variate map defined on an abstract choice set, is introduced. Then, and in a purely choice-theoretical framework, we relate the solutions of this equation to the notion of a *rationalizable agreement* rule. Finally, some results about the specific functional form of the solutions of the consensus functional equation in the setup of monetary payoffs and lotteries, over two outcomes, are also shown.

JEL Classification: C60, D63

Keywords: consensus equation, rationalizable agreement rules

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1 Introduction

The majority of economic situations can be described, in essence, in terms of an agreement between two individuals over a choice set, e.g., when this choice set is a space of monetary payoffs or a lottery-space over two outcomes. In this way, the provision of a formalization of the act of agreement between two individuals is of fundamental relevance for understanding the economic mechanisms. This paper provides such a formalization of the agreement, leading to the concept of the *consensus functional equation*, by supposing that $X \neq \emptyset$ is the choice set, the same for both individuals, and $F : X^2 = X \times X \to X$ is the map that expresses the agreement between them. In words, if individual 1 chooses alternative $x \in X$ while individual 2 chooses alternative $y \in X$, then $F(x, y)$ is the alternative in X that the two agents agree. The consensus equation is based upon the following simple idea. If an agreement is reached between the two individuals, then this agreement should be sufficiently robust that if either of the individiuals changes her/his initial position on the one agreed by both, then the former agreement should not be changed. In formula, $F(F(x, y), y) = F(x, F(x, y)) = F(F(x, y), F(x, y)) = F(x, y)$, for all $x, y \in X$. This is what we call the *consensus functional equation* (shortly, the *consensus equation*). It should be observed that the last equality of the formula is exactly the *unanimity principle* over the alternatives that are in the codomain of F (i.e., $F(z, z) = z$, for every $z \in F(X^2)$). Thus, if F satisfies *non-imposition* (i.e., for every $z \in X$ there are $x, y \in X$ such that $F(x, y) = z$), then the consensus equation implies the unanimity principle. The consensus equation transpires a property that, in some sense, recalls the Nash equilibrium concept of Game Theory. Indeed, if, for a given $x, y \in X$, $F(x, y)$ can be understood as the "best social agreement" (provided that agent 1 chooses alternative x whereas agent 2 chooses alternative y), then the "best choice" for agent 1, provided that agent 2 maintains her/his choice y, in order to reach this "best social agreement", is to single out $F(x, y)$. The same argument applies for agent 2. In addition, and although all of these interpretations of the consensus equation are made within the agreement framework, the ideas that it conveys can be directly applied to social choice theory context.

The paper is organized as follows. Section 2 contains the basic background. In Section 3 we study the consensus equation from an abstract point of view. To that end, we introduce a key concept; namely, that of a *rationalizable bi-variate map*. Rationalizability, for a bi-variate map F, is a notion that resembles the one already introduced in the literature of single-valued choice functions (see [1],[2] and [7] or, more recently, [6]). Rationalizability here means that a particular binary relation, that we call the *revealed consensus relation*, describes, in some sense, F. A major, and at the same time appealing, difference between the two notions of rationalizability is that a single-valued choice function always sends a subset of X into a point of this subset, whereas a rationalizable bi-variate map applies $F(x, y)$ into a point of X, possibly distinct from x and y, for every $x, y \in X$.

The existence of the revealed consensus relation on X makes the interpretation of a "best choice" given above to be meaningful. We characterize bi-variate maps that satisfy the consensus equation plus the *anonymity principle* as those that are rationalizable. Moreover, if a property, stronger than the fulfilment of the consensus equation together with the unanimity and the anonymity principles, is demanded; namely, that of being *associative*, then the latter result significantly improves. If an agreement rule between two individuals $F : X^2 \to X$ is associative, then there is a partial order, say $\preceq$, defined on X such that $(X, \preceq)$ is a *semi-lattice* and $F(x, y)$ turns out to be the supremum, with respect to $\preceq$, of $\{x, y\}$, for every $x, y \in X$.

An *agreement rule* is a unanimous and anonymous bi-variate map from X^2 into X that, in addition, satisfies the consensus equation. Associativity (i.e., $F(F(x, y), z) = F(x, F(y, z))$, for every $x, y, z \in X$) is a slightly more demanding property than the fulfilment of the consensus equation and can be viewed as an *extension* property. That is, if F is associative then, for any finite number of agents, we can induce agreement rules based on it. In other words, associativity invites everyone "to join the party".

We also pay attention to the case where F is a *selector*, i.e., $F(x, y) \in \{x, y\}$, $x, y \in X$. In this case, and for obvious reasons, we will say that F satisfies the *independence of irrelevant alternatives condition*. Quite surprisingly, the independence of irrelevant alternatives condition turns out to be stronger than consensus.

In Section 4 we study several aspects of the solutions of the consensus equation in concrete scenarios. In particular, we pay attention to the case in which X is a space of monetary payoffs or a lottery-space over two outcomes. This allows us to identify X to a real interval. We add some additional conditions, like monotonicity (Paretian properties) and continuity, on F. Then we present both impossibility as well as possibility results. On the one hand, we prove that there is no strongly Paretian bi-variate map which satisfies consensus. On the other hand, we show that the only continuous agreement rules that satisfy the independence of irrelevant alternatives condition are the max and the min (i.e., those based upon the most and the least favoured individuals, respectively).

Throughout the paper we will focus on the consensus equation involving only two individuals. The generalization of this equation for more than two people is left for a future article.

2 Preliminaries

In what follows, the (nonempty) choice set (or the set of alternatives) will be denoted by X and $F : X^2 = X \times X \to X$ will be a bi-variate map defined on X.

Definition 2.1. The map F is said to satisfy:

(1) the *unanimity principle* if $F(x, x) = x$ for every $x \in X$.

(2) the *anonymity principle* if $F(x, y) = F(y, x)$ for every $x, y \in X$.

(3) the *consensus functional equation (shortly, consensus)* if $F(F(x,y),y) = F(x, F(x,y)) = F(F(x,y), F(x,y)) = F(x,y)$ for every $x, y \in X$.

(4) the *associativity axiom* if $F(x, F(y,z)) = F(F(x,y), z)$ for every $x, y, z \in X$.

(5) the *independence of irrelevant alternatives condition (shortly, IIA))* if $F(x,y) \in \{x, y\}$, for every $x, y \in X$.

Definition 2.2. A bi-variate map $F : X^2 \to X$ is said to be an *agreement rule* if it satisfies the conditions (1) to (3) of Definition 2.1 above.

Now, we recall some basic concepts on binary relations. A binary relation $\preceq$ defined on X is said to be a *partial order* if it is reflexive (i.e., $x \preceq x$, for every $x \in X$), antisymmetric (i.e., $x \preceq y$ and $y \preceq x$ imply $x = y$, for every $x, y \in X$) and transitive (i.e., $x \preceq y$, $y \preceq z$ imply $x \preceq z$, for every $x, y, z \in X$). If, in addition, $\preceq$ is total (i.e., either $x \preceq y$ or $y \preceq x$, for every $x, y \in X$), then $\preceq$ is said to be a *total order*.

A binary relation, $\mathcal{R}$, on X is said to have the *supremum property* if, for every $x, y \in X$, there is a unique $z \in X$ such that the following two conditions are met: (i) $x\mathcal{R}z$ and $y\mathcal{R}z$. (ii) If there is $u \in X$ such that $x\mathcal{R}u$ and $y\mathcal{R}u$, then $z\mathcal{R}u$. The unique element z that satisfies conditions (i) and (ii) is called the *supremum* of x and y and it is denoted by $sup_{\mathcal{R}}\{x, y\}$. If, $sup_{\mathcal{R}}\{x, y\} \in \{x, y\}$, then it is called *maximum* of x and y and it is denoted by $max_{\mathcal{R}}\{x, y\}$.

Definition 2.3. Let $\preceq$ be a partial order defined on X. Then $(X, \preceq)$ is said to be a *semi-lattice* if $\preceq$ has the supremum property.[1]

3 Consensus vs. rationalizable bi-variate maps

The main purpose of this section is to provide a description of the agreement rules defined on X in terms of certain binary relations on X with special features. To that end, the following concept will play an important role.

Definition 3.1. Let F be a bi-variate map defined on X. Associated with F we can define the binary relation, denoted by $\mathcal{R}_{rc}$, on X as follows: $x\mathcal{R}_{rc}y \iff F(x,y) = y$, for every $x, y \in X$. We will say $\mathcal{R}_{rc}$ to be the *revealed consensus relation* of F.

Before introducing the notion of a rationalizable bi-variate map, a notational convention is needed.

Notation. Let $\mathcal{R}$ be a binary relation defined on X. Then, for each $x \in X$, $G_{\mathcal{R}}(x)$ will denote the *upper contour set* of x, i.e., $G_{\mathcal{R}}(x) = \{z \in X : x\mathcal{R}z\}$.

Definition 3.2. A bi-variate map F on X is said to be *rationalizable* if $F(x,y) \in G_{\mathcal{R}_{rc}}(x) \cap G_{\mathcal{R}_{rc}}(y)$, for every $x, y \in X$.

[1]For an excellent account of the material related to latticial or semi-latticial structures, see e.g., [3].

We now establish the first result of this section. It turns out to be a characterization of the bi-variate anonymous maps that satisfy the consensus equation in terms of those which are rationalizable.

Theorem 3.3. *Let F be a unanimous and anonymous bi-variate map defined on X. Then F is rationalizable if and only if it satisfies consensus.*

Proof. Suppose that F is an anonymous bi-variate map defined on X which satisfies the consensus equation. Let $x, y \in X$ be fixed. In order to show that F is rationalizable, notice that $F(x, F(x, y)) = F(x, y)$ since F satisfies the consensus equation. Thus, by definition of $\mathcal{R}_{rc}$, $F(x, y) \in G_{\mathcal{R}_{rc}}(x)$. Moreover, by anonymity together with consensus, it holds that $F(y, F(x, y)) = F(F(x, y), y) = F(x, y)$. Therefore, $F(x, y) \in G_{\mathcal{R}_{rc}}(y)$. So, $F(x, y) \in G_{\mathcal{R}_{rc}}(x) \cap G_{\mathcal{R}_{rc}}(y)$. Since x, y are arbitrary elements of X, it follows that F is rationalizable.

For the converse, suppose that F is an anonymous rationalizable bi-variate map defined on X. We want to see that F satisfies consensus. To that end, let $x, y \in X$ be fixed. Since F is rationalizable, it holds that $x\mathcal{R}_{rc}F(x, y)$ and $y\mathcal{R}_{rc}F(x, y)$. But, by definition of the revealed consensus relation, this means that $F(x, F(x, y)) = F(x, y)$ and $F(y, F(x, y)) = F(x, y)$. Now, by anonymity, $F(y, F(x, y)) = F(F(x, y), y)$ and therefore $F(F(x, y), y) = F(x, y)$. The fact that $F(F(x, y), F(x, y)) = F(x, y)$ follows directly from unanimity. Since x, y are arbitrary elements of X, we have shown that F satisfies consensus. $\square$

Remarks 3.4. (i) For a bi-variate map F, rationalizable or not, there could exist a binary relation $\mathcal{R}$ on X, other than $\mathcal{R}_{rc}$ if F is rationalizable, for which $F(x, y) \in G_{\mathcal{R}}(x) \cap G_{\mathcal{R}}(y)$, for every $x, y \in X$. If this is the case, then $\mathcal{R}_{rc} \subseteq \mathcal{R}$. In other words, $\mathcal{R}_{rc}$ is the coarser binary relation on X that satisfies the latter property. Indeed, let $\mathcal{R}$ be a binary relation defined on X so that $F(x, y) \in G_{\mathcal{R}}(x) \cap G_{\mathcal{R}}(y)$, for every $x, y \in X$. Let $x, y \in X$ arbitrary elements of X such that $x\mathcal{R}_{rc}y$. Then, by definition of $\mathcal{R}_{rc}$, we have that $F(x, y) = y$. Also, $F(x, y) \in G_{\mathcal{R}}(x) \cap G_{\mathcal{R}}(y)$. So, in particular, $y = F(x, y) \in G_{\mathcal{R}}(x)$ which means that $x\mathcal{R}y$. Therefore, $\mathcal{R}_{rc} \subseteq \mathcal{R}$.

(ii) A unanimous and anonymous bi-variate map F defined on X for which $\mathcal{R}_{rc}$ is transitive need not be rationalizable. Indeed, let $X = \{x, y, z\}$ and define $F : X \times X \to X$ as follows: $F(x, x) = F(y, z) = F(z, y) = x$, $F(y, y) = F(x, z) = F(z, x) = y$ and $F(z, z) = F(x, y) = F(y, x) = z$. Obviously, F, so-defined, is unanimous and anonymous. Then, an easy calculation gives: $\mathcal{R}_{rc} = \{(x, x), (y, y), (z, z)\}$. In other words, $x\mathcal{R}_{rc}x$, $y\mathcal{R}_{rc}y$ and $z\mathcal{R}_{rc}z$ are the only relations, according to $\mathcal{R}_{rc}$, among the three elements of X. Thus, in addition to being reflexive and antisymmetric, $\mathcal{R}_{rc}$ is transitive too. However, F is not rationalizable since, for example, $z = F(x, y) \notin G_{\mathcal{R}_{rc}}(x) \cap G_{\mathcal{R}_{rc}}(y)$.

In general, as the next proposition shows, for a (unanimous) bi-variate map F, consensus is a condition weaker than associativity or IIA.

Proposition 3.5. *Let F be a bi-variate map defined on X.*

(i) If F is unanimous and associative, then it safisfies consensus.

(ii) If F satisfies IIA, then it safisfies consensus.

Proof. (i) Let $x, y \in X$ be fixed. Then, by associativity and unanimity, it holds that $F(F(x,y),y) = F(x, F(y,y)) = F(x,y)$. The other equality of consensus is proved similarly. So, since x, y are arbitrary points of X, F satisfies consensus.

(ii) Let $x, y \in X$ be fixed. Since F satisfies IIA, either $F(x,y) = x$ or $F(x,y) = y$. If $F(x,y) = x$, then we have that $F(F(x,y),y) = F(x,y) = x = F(x,x) = F(x, F(x,y))$. Now, if $F(x,y) = y$, then $F(F(x,y),y) = F(y,y) = y = F(x,y) = F(x, F(x,y))$. So, in any of the two cases, we have that $F(F(x,y),y) = F(x, F(y,y)) = F(x,y)$. Since x, y are arbitrary points of X, F satisfies consensus. $\square$

As a direct consequence of Theorem 3.3 and Proposition 3.5 we obtain the following corollary.

Corollary 3.6. *(i) Every unanimous, anonymous and associative bi-variate map defined on X is rationalizable.*

(ii) Every bi-variate map defined on X which satisfies IIA is rationalizable.

Remark 3.7. It is simple to see that, for a unanimous and anonymous bi-variate map F, associativity and IIA are independent conditions. Moreover, there are agreement rules (hence rationalizable bi-variate maps) other than associative maps or those that satisfy IIA. For a thorough description of the links that can be established among the mentioned properties of bi-variate maps, see [5].

We now focus on associative agreement rules. As we have just seen, associativity is a more demanding property than consensus. Actually, and as we will see next, associativity strengthens, in a significant manner, the content of Theorem 3.3.

Theorem 3.8. *Let F be an associative agreement rule defined on X. Then, $(X, \mathcal{R}_{rc})$ is a semi-lattice and $F(x,y) = sup_{\mathcal{R}_{rc}}\{x,y\}$, for every $x, y \in X$.*

Proof. Let us first prove that $\mathcal{R}_{rc}$ is a partial order on X. Indeed, reflexivity follows directly from unanimity of F. To see that $\mathcal{R}_{rc}$ is antisymmetric, let $x, y \in X$ such that $x\mathcal{R}_{rc}y$ and $y\mathcal{R}_{rc}x$. Then, by definition of $\mathcal{R}_{rc}$, we have that $F(x,y) = y$ and $F(y,x) = x$. So, by anonymity, $x = y$ and therefore $\mathcal{R}_{rc}$ is antisymmetric. To prove transitivity of $\mathcal{R}_{rc}$, let $x, y, z \in X$ such that $x\mathcal{R}_{rc}y$, $y\mathcal{R}_{rc}z$. Then, by definition of $\mathcal{R}_{rc}$ again, we have that $F(x,y) = y$ and $F(y,z) = z$. Let us see that $F(x,z) = z$, which would mean that $x\mathcal{R}_{rc}z$. Indeed, $F(x,z) = F(x, F(y,z)) = F(F(x,y),z) = F(y,z) = z$, the second equality being true since F is associative. Thus, $\mathcal{R}_{rc}$ is transitive too.

Let us show now that $(X, \mathcal{R}_{rc})$ is a semi-lattice. To that end, we have to prove that, for given arbitrary elements $x, y \in X$, there is $sup_{\mathcal{R}_{rc}}\{x,y\}$. Notice that, since F is asociative, by Proposition 3.5(i), it satisfies consensus too. So, $F(x, F(x,y)) = F(x,y)$ and therefore, by definition of $\mathcal{R}_{rc}$, $x\mathcal{R}_{rc}F(x,y)$. In a similar way, now using anonymity and consensus, we have that $F(y, F(x,y)) =$

$F(F(x,y),y) = F(x,y)$. That is, $y\mathcal{R}_{rc}F(x,y)$. So $F(x,y)$ is an upper bound, with respect to $\mathcal{R}_{rc}$, of x and y. Let us see that it is the least upper bound. For that, let $z \in X$ such that $x\mathcal{R}_{rc}z$ and $y\mathcal{R}_{rc}z$. Then, by definition of $\mathcal{R}_{rc}$ again, we have that $F(x,z) = F(y,z) = z$. So, $F(F(x,y),z)) = F(x,F(y,z)),y) = F(x,z) = z$, the first equality being true by associativity. Therefore, we have that $F(F(x,y),z)) = z$ which means that $F(x,y)\mathcal{R}_{rc}z$. So, we have shown that $F(x,y) = sup_{\mathcal{R}_{rc}}\{x,y\}$, which proves the second assertion of the statement of Theorem 3.8 and the proof is complete. □

We now present some illuminating observations about the concepts introduced above.

Remarks 3.9. (i) It should be observed that, if $\mathcal{R}$ is a binary relation on X for which $(X,\mathcal{R})$ is a semi-lattice, then the bi-variate map defined as $(x,y) \in X \times X \rightsquigarrow F_{\mathcal{R}}(x,y) = sup_{\mathcal{R}}\{x,y\} \in X$ is an associative agreement rule. Moreover, in this case, it can be easily proved that $\mathcal{R} \equiv \mathcal{R}_{rc}$. So, associative agreement rules are characterized as those that can be rationalized by means of semi-latticial structures.

(ii) An agreement rule that satisfies IIA need not be associative. Moreover, and unlike the associative case, the revealed consensus relation $\mathcal{R}_{rc}$ in this situation can exhibit intransitivities. To see an example, consider the set $X = \{x,y,z\}$ and the bi-variate map $F : X \times X \to X$ given by $F(x,x) = F(x,z) = F(z,x) = x$; $F(x,y) = F(y,x) = F(y,y) = y$; $F(y,z) = F(z,y) = F(z,z) = z$. It is clear that F is anonymous and satisfies IIA. However, it is not associative since $F(x,F(y,z)) = F(x,z) = x$, whereas $F(F(x,y),z) = F(y,z) = y$. In terms of the revealed consensus relation $\mathcal{R}_{rc}$ we have that $x\mathcal{R}_{rc}y$, $y\mathcal{R}_{rc}z$ and $z\mathcal{R}_{rc}x$. So, there is a "cycle", with respect to $\mathcal{R}_{rc}$, for the three-element set $\{x,y,z\}$.

(iii) If an agreement rule F satisfies IIA, then the revealed consensus relation $\mathcal{R}_{rc}$ becomes a total order on X. Moreover, if an agreement rule F which satisfies IIA is also associative, then $F(x,y) = max_{\mathcal{R}_{rc}}\{x,y\}$, for every $x,y \in X$. So, associative agreement rules that satisfy IIA are characterized as those that can be rationalized by means of totally ordered structures.

(iv) Associative agreement rules have an interesting property that we call the *extension property*. The extension property means that an associative (bi-variate) agreement rule "generates" associative, unanimous and anonymous n-variate rules, for any finite number of agents $n \in \mathbb{N}$. In words, if a (unanimous and anonymous) map involving just two individuals is associative then it is possible that more and more individuals can "join the party" and enjoy a "stable" agreement. So, from a behavioural perspective, associativity is an appealing property. Indeed, let F_2 be an associative (bi-variate) agreement rule. Then, by Theorem 3.8, $F_2(x,y) = sup_{\mathcal{R}_{rc}}\{x,y\}$, for every $x,y \in X$. Now, for any $n \geq 3$, define $F_n : X^n = X \times \ldots \times X_{(n-times)} \to X$ as follows: $F_n(x_1,\ldots,x_n) = sup_{\mathcal{R}_{rc}}\{x_1,\ldots,x_n\}$, for every $x_1,\ldots,x_n \in X$. It is then simple to see that, for every $n \geq 3$, F_n so-defined is an associative, unanimous and anonymous n-variate map.

(v) It should be noted that Theorem 3.8 can be applied to scenarios in which the choice set X is, on its own, a space of preferences. Indeed, let $X = \{\precsim \subseteq Z \times Z\,; \precsim$ is a transitive and total binary relation defined on a finite set $Z\}$. Let $F : X \times X \to X$ be the *Borda rule* (see, [6]). Then, it is straightforward to see that F is an associative agreement rule. So, Theorem 3.8 says that the Borda rule is entirely described by the revealed consensus relation on X. Actually, it is simple to prove that, in this case, $\mathcal{R}_{rc}$ is given by: $\precsim_1 \mathcal{R}_{rc} \precsim_2$ if and only if $\precsim_2 \subseteq \precsim_1$ and $\prec_1 \subset \prec_2$, $(\precsim_1, \precsim_2 \in X)$. Here, $\prec$ stands for the asymmetric part of $\precsim$ (i.e., $x \prec y$ if and only if $\neg(y \precsim x)$, for every $x, y \in X$).

As seen in the proof of Theorem 3.8, an associative, unanimous and anonymous bi-variate map defined on X has the property that its revealed consensus relation turns out to be transitive. The converse is not true even though the bi-variate map is rationalizable (or, equivalently by Theorem 3.3, it satisfies consensus). Nevertheless, for a unanimous bi-variate map that satisfies IIA, transitivity of its revealed consensus relation implies associativity. These two facts are the content of the next result which concludes this section.

Proposition 3.10. *(i) An agreement rule such that its associated revealed consensus relation is transitive need not be associative.*

(ii) Every agreement rule that satisfies IIA is associative.

Proof. (i) Let $X = \{x, y, z, u\}$. Let $F : X \times X \to X$ be the bi-variate map given by $F(x, x) = x$; $F(y, y) = y$; $F(x, y) = F(x, z) = F(y, z) = F(y, z) = F(z, x) = F(z, y) = F(z, z) = F(z, u) = F(u, z) = z$; $F(x, u) = F(y, u) = F(u, x) = F(u, y) = F(u, u) = u$. It is clear that F satisfies unanimity and anonymity. Let us see that it is an agreement rule (i.e., it satisfies consensus) by showing that it is rationalizable (see, Theorem 3.3). To that end, let $\mathcal{R}_{rc}$ be its revealed consensus relation. A direct calculation proves that $\mathcal{R}_{rc}$ is given by: $x\mathcal{R}_{rc}x$, $x\mathcal{R}_{rc}z$, $x\mathcal{R}_{rc}u$, $y\mathcal{R}_{rc}y$; $y\mathcal{R}_{rc}z$, $y\mathcal{R}_{rc}u$, $z\mathcal{R}_{rc}z$, $u\mathcal{R}_{rc}z$, $u\mathcal{R}_{rc}u$. Let us observe that $\mathcal{R}_{rc}$ is transitive. Now, by checking the upper contour sets of $\mathcal{R}_{rc}$, we obtain: $G_{\mathcal{R}_{rc}}(x) = \{x, z, u\}$; $G_{\mathcal{R}_{rc}}(y) = \{y, z, u\}$; $G_{\mathcal{R}_{rc}}(z) = \{z\}$; $G_{\mathcal{R}_{rc}}(u) = \{z, u\}$. Thus $F(x, x) = x \in G_{\mathcal{R}_{rc}}(x)$; $F(x, y) = z \in G_{\mathcal{R}_{rc}}(x) \cap G_{\mathcal{R}_{rc}}(y)$; $F(x, z) = z \in G_{\mathcal{R}_{rc}}(x) \cap G_{\mathcal{R}_{rc}}(z)$; $F(x, u) = u \in G_{\mathcal{R}_{rc}}(x) \cap G_{\mathcal{R}_{rc}}(u)$; $F(y, y) = y \in G_{\mathcal{R}_{rc}}(y)$; $F(y, z) = z \in G_{\mathcal{R}_{rc}}(y) \cap G_{\mathcal{R}_{rc}}(z)$; $F(y, u) = u \in G_{\mathcal{R}_{rc}}(y) \cap G_{\mathcal{R}_{rc}}(u)$; $F(z, z) = z \in G_{\mathcal{R}_{rc}}(z)$; $F(z, u) = z \in G_{\mathcal{R}_{rc}}(z) \cap G_{\mathcal{R}_{rc}}(u)$. Therefore F is rationalizable. Finally, observe that F is not associative since $F(F(x, y), u) = F(z, u) = z \neq u = F(x, u) = F(x, F(y, u))$.
(ii) Let $x, y, z \in X$ be fixed. We have to show that $F(F(x, y), z) = F(x, F(y, z))$. Since F satisfies IIA $F(x, y) = \{x, y\}$, $F(x, z) = \{x, z\}$ and $F(y, z) = \{y, z\}$. So we distinguish among eight possibilities:
(1) $F(x, y) = x$, $F(x, z) = x$ and $F(y, z) = y$. In this case, $F(F(x, y), z) = F(x, z) = x = F(x, y) = F(x, F(y, z))$ and we are done.
(2) $F(x, y) = x$, $F(x, z) = x$ and $F(y, z) = z$. In this case, $F(F(x, y), z) = F(x, z) = x = F(x, z) = F(x, F(y, z))$ and we are done again.
(3) $F(x, y) = x$, $F(x, z) = z$ and $F(y, z) = y$. Now, since F is anonymous, $F(y, x) = F(x, y) = x$ and $F(z, y) = F(y, z) = y$. So, $y\mathcal{R}_{rc}x$ and $x\mathcal{R}_{rc}z$ which,

by transitivity of $\mathcal{R}_{rc}$, implies that $y\mathcal{R}_{rc}z$. But $F(z,y) = F(y,z) = y$ means that $z\mathcal{R}_{rc}y$ too. Now, $\mathcal{R}_{rc}$ is antisymmetric since F is anonymous. Therefore, $y = z$. Now, if $y = z$, $F(F(x,y),z) = F(x,F(y,z))$ becomes $F(F(x,y),y) = F(x,F(y,y))$ or, equivalently, $F(F(x,y),y) = x = F(x,y) = F(x,F(y,y))$ and we are done.
(4) $F(x,y) = x$, $F(x,z) = z$ and $F(y,z) = z$. In this case, $F(F(x,y),z) = F(x,z) = z = F(x,F(y,z))$ and we are done.
(5) $F(x,y) = y$, $F(x,z) = x$ and $F(y,z) = y$. In this case, $F(F(x,y),z) = F(y,z) = y = F(x,y) = F(x,F(y,z))$ and we are done.
(6) $F(x,y) = y$, $F(x,z) = x$ and $F(y,z) = z$. In this case, and arguing in the same way as in case (3) above, we have that $x\mathcal{R}_{rc}y$ and $z\mathcal{R}_{rc}x$ which, by transitivity, implies that $z\mathcal{R}_{rc}y$. This, together with $y\mathcal{R}_{rc}z$, implies that $y = z$. Then, $F(F(x,y),z) = F(x,F(y,z))$ becomes $F(F(x,y),y) = F(x,F(y,y))$ or, equivalently, $F(F(x,y),y) = y = F(x,y) = F(x,F(y,y))$ and we are done again.
(7) $F(x,y) = y$, $F(x,z) = z$ and $F(y,z) = y$. In this case, $F(F(x,y),z) = F(y,z) = y = F(x,y) = F(x,F(y,z))$ and we are done. Finally,
(8) $F(x,y) = y$, $F(x,z) = z$ and $F(y,z) = z$. In this case, $F(F(x,y),z) = F(y,z) = z = F(x,z) = F(x,F(y,z))$ which concludes the proof. □

Remark 3.11. It can be shown that if X is a three-elements set (i.e, $X = \{x,y,z\}$), then any agreement rule defined on X for which $\mathcal{R}_{rc}$ is transitive is, in fact, associative.

4 Possibility vs. impossibility results in continuum spaces

In this section, we study the consensus equation in particular contexts. In general, this equation has no easy solutions (see [5] for details). However, in concrete spaces, and with some natural additional assumptions on the map F, it is possible to entirely describe its solutions. We assume that the choice set X is a real interval. In particular, this means that X can be a set of monetary payoffs or, in a probabilistic scenario, X could represent the space of lotteries between two outcomes. In the first situation X can be identified as $[0,\infty)$ whereas it can be identified as $[0,1]$ in the second. Both impossibility as well as possibility results arise. On the one hand, we prove that there is no strongly Paretian bi-variate map which satisfies consensus. On the other hand, we show that the only continuous agreement rules that satisfy IIA are the max and the min. In what follows, $\mathcal{I}$ will represent an interval of the real line $\mathbb{R}$.

Notation. Let $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ be a bi-variate map. For every $x \in \mathcal{I}$, F_x (respectively, F^x) stands for the *vertical* (respectively, *horizontal*) *restriction* of F; i.e., $y \in \mathcal{I} \rightsquigarrow F_x(y) = F(x,y) \in \mathcal{I}$ (respectively, $y \in \mathcal{I} \rightsquigarrow F^x(y) = F(y,x) \in \mathcal{I}$).

We recall the concept of an idempotent function defined on $\mathcal{I}$. This concept will play a significant role in the sequel, in particular in the next Proposition 4.2.

Definition 4.1. A function $f : \mathcal{I} \to \mathcal{I}$ is said to be *idempotent* if $f(f(x)) = f(x)$, for every $x \in \mathcal{I}$.

Proposition 4.2. *Let $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ be a bi-variate map.*

(i) F is unanimous if and only if $F_x(x) = F^x(x) = x$ for every $x \in \mathcal{I}$.

(ii) F is anonymous if and only if $F_x(y) = F^x(y)$ for every $x, y \in \mathcal{I}$.

(iii) F satisfies consensus if and only if, for every $x \in X$, both restrictions F_x and F^x are idempotent functions and, for each $z \in F(\mathcal{I} \times \mathcal{I})$, it holds that $F_z(z) = F^z(z) = z$.

Proof. Parts (i) and (ii) follow directly. So we prove only part (iii). Suppose that F satisfies consensus and let $x \in X$ be fixed. Then, we have that $F_x(F_x(y)) = F(x, F_x(y)) = F(x, F(x, y)) = F(x, y) = F_x(y)$ for every $y \in \mathcal{I}$. Also, we have that $F^x(F^x(y)) = F(F^x(y), x) = F(F(y, x), x) = F(y, x) = F^x(y)$, for every $y \in \mathcal{I}$. Therefore, F_x and F^x are both idempotent functions. Since x is an arbitrary element of $\mathcal{I}$, we have proven that F_x and F^x are both idempotent functions foe every $x \in \mathcal{I}$. The fact that, for each $z \in F(\mathcal{I} \times \mathcal{I})$, $F_z(z) = F^z(z) = z$ follows directly from consensus.

Conversely, suppose that, for every $x \in \mathcal{I}$, F_x and F^x are both idempotent functions. Let $x, y \in \mathcal{I}$ be fixed. Then, we have that $F(x, F(x, y)) = F_x(F_x(y)) = F_x(y) = F(x, y)$, and also we have that $F(x, y) = F^y(x) = F^y(F^y(x)) = F(F^y(x), y) = F(F(x, y), y)$. Moreover, $F(F(x, y), F(x, y)) = F(x, y)$ since, by hypothesis, $F_z(z) = F^z(z) = z$, for every $z \in F(\mathcal{I} \times \mathcal{I})$. Therefore, F satisfies consensus. □

Remark 4.3. It should be noted that the concepts introduced above can be given in an abstract space X. In particular, Proposition 4.2 remains true if $\mathcal{I}$ is replaced by a nonempty choice set X.

Before presenting a basic definition of the most familiar notions involving monotonicity properties of real-valued bi-variate functions, we recall that for a given $(x, y), (u, v) \in \mathcal{I} \times \mathcal{I}$, $(x, y) \le (u, v)$ means $x \le u$ and $y \le v$; $(x, y) < (u, v)$ means $(x, y) \le (u, v)$ and $(x, y) \ne (u, v)$. Finally, $(x, y) \ll (u, v)$ means $x < u$ and $y < v$.

Definition 4.4. A bi-variate map $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ is said to be:

(i) *Paretian (or, non-decreasing)* if $(x, y) \le (u, v)$ implies $F(x, y) \le F(u, v)$, for every $x, y, u, v \in \mathcal{I}$.

(ii) *weakly Paretian* if $(x, y) \ll (u, v)$ implies $F(x, y) < F(u, v)$, for every $x, y, u, v \in \mathcal{I}$.

(iii) *strongly Paretian* if $(x, y) < (u, v)$ implies $F(x, y) < F(u, v)$, for every $x, y, u, v \in \mathcal{I}$.

(iv) *dictatorial* if either $F(x, y) = x$ for every $x, y \in \mathcal{I}$, or $F(x, y) = y$ for every $x, y \in \mathcal{I}$.

(v) *continuous* if the inverse image of every Euclidean open subset of $\mathcal{I}$ is an open subset of $\mathcal{I} \times \mathcal{I}$, where $\mathcal{I} \times \mathcal{I}$ is endowed with the usual product (Euclidean) topology.

Notice that the monotonicity properties that appear in Definition 4.4 above are meaningful in the case that the choice set X is a set of monetary payoffs. Now we present an impossibility result. It states that, for a bi-variate map on $\mathcal{I}$, strongly Paretian and consensus are incompatible conditions.

Theorem 4.5. *There is no strongly Paretian bi-variate map on $\mathcal{I}$ which satisfies consensus.*

Proof. Suppose that $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ is a strongly Paretian bi-variate map which satisfies consensus. Let $x_0 \in \mathcal{I}$ be fixed and consider the vertical restriction F_{x_0}. Notice that F_{x_0} is a strictly increasing function since F is, by hypothesis, strongly Paretian. Let us see that $F_{x_0}(y) = y$, for all $y \in \mathcal{I}$ (in other words, F_{x_0} is the identity function from $\mathcal{I}$ into $\mathcal{I}$). To that end, suppose, by way of contradiction, that there is $y_0 \in \mathcal{I}$ such that $F_{x_0}(y_0) \neq y_0$. Then, either $F_{x_0}(y_0) < y_0$, or $F_{x_0}(y_0) > y_0$. Suppose that the first case occurs. Then, since F_{x_0} is a strictly increasing real-valued function, $F_{x_0}(F_{x_0}(y_0)) < F_{x_0}(y_0)$. But this leads to a contradiction since, by Proposition 4.2 (iii), F_{x_0} is an idempotent function and therefore $F_{x_0}(F_{x_0}(y_0)) = F_{x_0}(y_0)$. So, $F_{x_0}(y_0) < y_0$ cannot occur. In the same way we can prove that $F_{x_0}(y_0) > y_0$ cannot happen for any $y_0 \in \mathcal{I}$. Therefore, we have proved that $F_{x_0}(y) = y$, for every $y \in \mathcal{I}$. Actually, since x_0 is an arbitrary element of $\mathcal{I}$, we have shown that, for every $x \in \mathcal{I}$, $F_x(y) = y$, for all $y \in \mathcal{I}$.

Arguing in a similar way we can prove that, for every $x \in \mathcal{I}$, the horizontal restriction F^x is the identity function. In other words, we have that, for every $x \in \mathcal{I}$, $F^x(y) = y$, for all $y \in \mathcal{I}$. Let us now reach a contradiction. Indeed, let $x \neq y$ be two elements in $\mathcal{I}$. Then, $F(x, y) = F_x(y) = y \neq x = F^y(x) = F(x, y)$, which is impossible. So, no strongly Paretian bi-variate map F which satisfies consensus can exist. □

Remarks 4.6. (i) A careful glance at the proof of Theorem 4.5 above shows that the only bi-variate map on $\mathcal{I}$ which satisfies consensus and has the additional property that all of its vertical restrictions are strictly increasing functions (respectively, all of its horizontal restrictions are strictly increasing functions) is dictatorial over the second (respectively, first) coordinate. That is, $F(x, y) = y$ for every $x, y \in \mathcal{I}$ (respectively, $F(x, y) = x$ for every $x, y \in \mathcal{I}$).

(ii) If strongly Paretian is relaxed to Paretian (or weakly Paretian) then the impossibility result does not hold true. For example, consider the dictatorial bi-variate maps or the max/min functions.

We now state a possibility result. Actually, we establish that the only continuous bi-variate maps $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ that satisfy IIA are the max, the min and the dictatorial functions. In particular, we have that the only continuous agreement rules that satisfy IIA are the max and the min functions.

Theorem 4.7. *Let $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ be a bi-variate map. Then the following conditions are equivalent:*

(i) F is continuous and satisfies IIA.

(ii) F is of one of the following forms:
(1) $F(x, y) = x$, for every $x, y \in \mathcal{I}$.
(2) $F(x, y) = y$, for every $x, y \in \mathcal{I}$.
(3) $F(x, y) = max\{x, y\}$, for every $x, y \in \mathcal{I}$.
(4) $F(x, y) = min\{x, y\}$, for every $x, y \in \mathcal{I}$.

Proof. (ii) implies (i) is straightforward.

For the converse, (i) implies (ii), let $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ be a continuous bi-variate map which satisfies IIA. Let $x \in \mathcal{I}$ be fixed and consider the vertical restriction F_x. Since F satisfies IIA, $F_x(y) \in \{x, y\}$ for all $y \in \mathcal{I}$. The continuity of F_x, together with IIA, clearly implies that F_x must be of one of the following types:
(1) $F_x(y) = x$, for every $y \in \mathcal{I}$.
(2) $F_x(y) = y$, for every $y \in \mathcal{I}$.
(3) $F_x(y) = y$, if $y \geq x$ and $F_x(y) = x$, if $y < x$.
(4) $F_x(y) = x$, if $y \geq x$ and $F_x(y) = y$, if $y < x$.

Now, the continuity of F (in two variables) clearly implies that if for some $x_0 \in \mathcal{I}$, F_{x_0} is of the type (i), i= 1 to 4, then F_x is of the type (i), for all $x \in \mathcal{I}$. Finally, it is straightforward to see that the situation for each of the four cases leads to the corresponding functional form given in the statement of the theorem. □

As a direct consequence of Theorem 4.7 we obtain the following corollary.

Corollary 4.8. *Let $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ be a bi-variate map. Then the following conditions are equivalent:*

(i) F is continuous, anonymous and satisfies IIA.

(ii) Either $F(x, y) = max\{x, y\}$ (for all $x, y \in \mathcal{I}$), or $F(x, y) = min\{x, y\}$ (for all $x, y \in \mathcal{I}$).

Remark 4.9. The content of Theorem 4.7 strongly depends on the condition that F satisfies IIA since there are continuous bi-variate maps $F : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ which satisfy consensus other than those belonging to the four types that appear in the statement of the theorem (for details, see [5]).

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