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Comparative Statics for a Consumer with Possibly Multiple Optimum Consumption Bundles

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ABSTRACT

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Non-positivity of the generalized substitution effect, non-positivity of the own-price substitution effect, homogeneity of degree zero in all prices and income, and the law of demand are some of the most primitive comparative static results in the standard revealed preference theory of consumers' behaviour. These results are however derived for demand functions. The literature does not have corresponding comparative static results for the more plausible case of demand correspondences, where the consumer is permitted to have multiple chosen bundles in a given price-income situation. Using the revealed preference approach to the theory of consumers' behaviour, this note establishes such results for demand correspondences; the analysis can be readily adapted to prove corresponding results in the preference-based approach.

JEL Classification: D11

Keywords: demand correspondence, weak axiom of revealed preference, non-positivity of generalized substitution effect, non-positivity of own-price substitution effect, homogeneity of degree zero, law of demand

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1. Introduction

Non-positivity of the generalized substitution effect, non-positivity of the own-price substitution effect, homogeneity of degree zero in all prices and income, and the 'law' of demand' (that is, the 'law' that a fall in the price of a non-inferior commodity does not reduce the quantity purchased of it) are some of the most primitive results in the standard theory of consumers' behaviour.[1] These results are, however, derived for the special case where the consumer has exactly one chosen consumption bundle for each price-wealth situation. The literature does not appear to have corresponding comparative statics results when a consumers may have multiple chosen consumption bundles in a given price-wealth situation[2], so that there may be a demand correspondence but not a demand function for the consumer. In fact, despite the advances in methods of comparative statics (see, for instance, Milgrom and Roberts (1994)), which, among other things, suggest the possibility of interesting comparative static exercises in the presence of multiple optima, sometimes there seems to be a preconception that comparative statics results cannot be derived when the consumer may have several optimum consumption bundles for the same price-income situation. Using the revealed preference approach, this note shows that it is indeed possible to derive comparative statics results for demand correspondences, which are exactly analogous to the familiar comparative statics results for demand functions. The latter turn out to be special cases of our more general results. Though we have chosen to derive our results in the revealed prefernce framework, our basic analysis can be adapted to derive exactly identical comparative static results for demand correspondences in the preference-based approach to the theory of consumer' behavior. Thus, irrespective of whether one chooses to use the revealed preference approach or the preference-based approach to analyse the behavior of consumers, the analysis need not be confined to the intuitively and empirically restrictive framework of demand functions to derive meaningful and testable comparative static conclusions.

2. The notation

Let $\Re^n_+$ and $\Re^n_{++}$, respectively, denote the set all non-negative real numbers and the set of all positive real numbers. Let *n* be the number of commodities. We assume that $\Re^n_+$ is the

[1] See, for example, Mas-Colell *et al.* (1995, pp.28-36) for a discussion.

[2] This is so despite the fact that multiple-element choice sets are permitted in many important contributions to the revealed preference literature specifically on the theory of consumer's choice (see, for example, Richter (1966)) as well as many important contributions to the revealed preference literature on the theory of choice in general (see, for example, Arrow (1959) and Sen (1971)).

consumer's consumption set. The commodity bundles will be denoted by x, x', etc. Given a consumption bundle x, x_i will denote the amount of the *i*-th commodity contained in *x*.

Prices will be assumed to be strictly positive. $\Re^n_{++}$ is the set of all possible price vectors. p, p', etc. will denote the price vectors. For any given commodity, i, we say that two price vectors, p and p' are *i-variants* iff $p_i \neq p_i^{'}$ and, for every commodity $j \neq i$, $p_j = p_i$. The income of the consumer will be denoted by $I, I', ... \in \Re_{++}$. A *price-income situation* will be denoted by (p, I), where $p \in \Re^n_{++}$ and $I \in \Re_{++}$. The set of all possible price-income situations will be denoted by Q.

Given a price-income situation (p, I), the consumer's *budget set*, denoted by $B(p, I)$, is defined to be the set of x in $\Re^n_+$ such that $I \geq p.x$

3. The basic concepts

A *demand correspondence* is a rule D which, for every $(p, I) \in Q$, specifies exactly one non-empty subset, $D(p, I)$, of $B(p, I)$. A *demand function* is a demand correspondence D such that, for all $(p, I) \in Q$, $D(p, I)$ contains exactly one commodity bundle.

A demand correspondence D satisfies *income-exhaustion* (IE) if and only if , for every price-wealth situation (p, I) and for every $x \in D(p, I)$, $p.x = I$.

Throughout this paper we assume that the demand correspondence satisfies income-exhaustion.

For every $(p, I) \in Q$, and every commodity i, $D_i(p, I)$ will denote the set of all $g \in \Re_+$, such that $x_i = g$ for some $x \in D(p, I)$ }. Thus, $D_i(p, I)$ is the set of the different quantities of commodity i that figure in the bundles in $D(p, I)$. Note that, by definition, 0 is a lower bound for $D_i(p, I)$ and $\frac{I}{p_i}$ is an upper bound for $D_i(p, I)$. Therefore, $D_i(p, I)$ will have an infimum and a supremum, which we shall denote by $\inf D_i(p, I)$ and $\sup D_i(p, I)$, respectively.

A commodity, i, is said to be *non-inferior* if and only if , for all $p \in \Re^n_{++}$ and all $I, I' \in \Re_+$, such that $I' > I$, $\inf D_i(p, I') \geq \inf D_i(p, I)$.

Thus, if i is a non-inferior good, it is not possible that, when the income increases, the price vector remaining the same, some quantity of commodity i that the consumer may buy in the new situation is strictly less than every quantity of commodity i that she might have bought in the initial situation, though it is possible that every quantity of commodity i that the consumer may buy in the new situation is less than some quantity of commodity i that she might have bought in the initial situation. In Figure 1, given the intial budget set oab corresponding to a price-wealth situation (p, I), suppose $D(p, I)$ is the set of all consumption bundles in the segment cd. Now suppose, we have a new price-wealth situation (p, I') such that $I' > I$, and the new budget set is $oa'b'$. The non-inferiority of commodity 1 will rule out the possibility that $D(p, I') = c'd'$ though it does not rule out the possibility that $D(p, I') = d'd''$.

Insert Figure 1

A demand correspondence D satisfies *the Weak Axiom of Revealed Preference* (WARP) iff, for all $(p, I), (p', I') \in Q$ and and all $x, x' \in \Re^n_+$, if [$x \in D(p, I)$ and $x' \in (B(p, I) - D(p, x))$], then not[$x, x' \in B(p', I')$ and $x' \in D(p', I'')$].

WARP, as defined above, is equivalent to Richter's (1966) Weak Congruence Axiom. Our WARP is identical to Sen's Weak Axiom of Revealed Preference except for the fact that Sen (1971) considers the problem of choice in general rather than the problem of choice by a competitive consumer. Note that the following axiom (we call it WARP') is closer to the original weak axiom of revealed preference due to Samuelson (1938):

WARP': For all $(p, I), (p', I') \in Q$ and all $x, x' \in \Re^n_+$, if [$x \in D(p, I)$ and $x' \in B(p, I)$ and $x \neq x'$], then not[$x, x' \in B(p', I')$ and $x' \in D(p', x')$].

If, however, we impose WARP' on the demand correspondence, the demand correspondence will become a demand function and that will defeat our basic purpose in this note.

Next we consider the notion of the substitution effect for demand correspondences. When we have a demand function, to define the substitution effect generated by a change in the price vector, we adjust the consumer's income in such a way that the consumer's income after the adjustment, together with the new price vector, is just enough to buy the initially chosen consumtion bundle. With a demand correspondence, the initially chosen consumption bundle may not be unique. Therefore, for different initially chosen bundles, we need to have different adjustments in the consumer's income after the change in the price vector, and we have to conssider the effect of each such adjustment.

The demand correspondence, D, satisfies *non-positivity of the generalized substituion effect* (NGSE) if and only if, for all $(p, I), (p', I') \in Q$ and all $x \in D(p, I)$ such that $I' = p'.x$, [for all

$x' \in D(p', I')$, $(p - p').(x - x') \leq 0$] and [for all $x' \in D(p', I')$ such that $x' \notin D(p, x)$, $(p - p').(x - x') < 0$]. It satisfies *non-positivity of the own-price substitution effect* (NOPSE) iff for all $(p, I), (p', I') \in Q$ and all $x \in D(p, I)$, such that [p and p' are i-variants, $p_i' < p_i$, and $I' = p'.x$], $\inf D_i(p', I') \geq x_i$.

When the demand correspondence is a demand function, NGSE and NOPSE are simply the corresponding classical Samuelson (1938) properties for demand functions.

The demand correspondence, D, satisfies *homogeneity of degree* 0 iff, for for all $(p, I) \in Q$ and all $\lambda \in \Re_{++}$, $D(p, I) = D(\lambda p, \lambda I)$.

4. Results

We now present our results. We first show that, given income exhaustion, the weak axiom of revealed preference and negativity of the generalized substitution effect are equivalent restrictions.

Proposition 1. *Suppose the demand correspondence D satisfies income-exhaustion. Then D satisfies non-positivity of the generalized substitution effect if and only if it satisfies the weak axiom of revealed preference.*

Proof: See the appendix.

Proposition 1 immediately yields the following.

Corollary 1. *Suppose the demand correspondence D satisfies income exhaustion and the weak axiom of revealed preference. Then D satisfies non-positivity of the own-price substitution effect.*

In light of our definition of a non-inferior commodity and Corollary 1, Proposition 1 also yields the following generalized version of the 'law of demand'.

Corollary 2. *Suppose the demand correspondence satisfies income exhaustion and the weak axiom of revealed preference. Further, suppose that commodity i is a non-inferior commodity. Then, for all $p, p' \in \Re^n_{++}$ such that [p and p' are $i-$variants and $p_i' < p_i$], and for all $I \in \Re_{++}$,* $\inf D_i(p', I) \geq \sup D_i(p, I)$.

Corollary 2 implies that if the price of a non-inferior commodity falls, the other prices and the income remaining the same, every quantity of the commodity that the consumer may buy in the new situation must be at least as great as every quantity of the commodity that the consumer may buy in the original situation.

Lastly, it turns out that, given income exhaustion, non-positivity of the generalized substitution effect implies homogeneity of degree 0. Proposition 1 thus additionally yields the following.

Corollary 3. *Suppose the demand correspondence satisfies income-exhaustion and the weak axiom of revealed preference. Then it must satisfy homogeneity of degree 0.*

Proof: See the appendix.

The traditional comparative static results in the demand function framework are evidently special cases of Proposition 1 and Corollaries 1, 2 and 3.

5. Concluding remarks

Using the revealed preference approach, we have derived for demand correspondences the counterparts of the very basic and familiar comparative static results for demand functions. While we have chosen to use the revealed preference framework, our analysis can be readily adapted to the preference-based framework. To avoid tedious repetition, we do not undertake that exercise here. It can, however, be easily checked that, by combining the notions of non-inferior goods, non-positivity of the generalized substitution effect, and non-negativity of the own-price substitution effect, which we have introduced in this paper, with the non-differential version of the preference-based theory of demand (see Yokoyama (1953) for an elegant exposition[3]), one can again derive the counterparts of the standard comparative static properties of demand functions.

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[3] See also, Hicks (1956).

Appendix

Proof of Proposition 1:

(i) First let D satisfy IE and WARP. We shall show that D must then satisfy NGSE. Consider $(p,I),(p',I')\in Q$ and $x\in D(p,I)$ such that $I'=p'.x$. Let $x'\in D(p',I')$. We then have $p'.x'\le I'=p'.x$, so that

$$p'.(x-x')\ge 0\,. \tag{1}$$

Now we shall show that

$$0\ge p.(x-x')\,, \tag{2}$$

and,

$$\text{if } x'\notin D(p,x)\text{, then } 0>p.(x-x')\,. \tag{3}$$

Suppose $p.x>p.x'$. Then, noting $x\in D(p,I)$ and IE, $p.x'<p.x=I$. Then, by IE, $x'\in(B(p,I)-D(p,I))$. Since $I'=p'.x$, $x\in B(p',I')$. Thus, we have $x\in D(p,I)$, $x'\in(B(p,I)-D(p,I))$, $x\in B(p',I')$, and $x'\in D(p',I')$. This contradicts WARP. Thus, (2) holds. Now suppose $x'\notin D(p,x)$. Then, given $x\in D(p,I)$, $I'=p'.x$ and $x'\in D(p',I')$, $p.x=p.x'$ will violate WARP. Noting (2), (3) follows immediately. Together, (1), (2) and (3) yield NGSE.

(ii) Now suppose D satisfies IE, but violates WARP. We shall show that NGSE must then be violated. Since WARP is violated, for some $(p,I),(p',I')\in Q$ and some $x,x'\in\Re^n_+$, $x\in D(p,I)$, $x'\in[B(p,I)$ - $D(p,I)$], $x,x'\in B(p',I')$, and $x'\in D(p',I')$]. Noting IE, it follows that:

$$p.x=I\ge p.x'\,, \tag{4}$$

and

$$p'.x'=I'\ge p'.x. \tag{5}$$

If $p'.x'=p'.x$, then, noting (4) and ($x\in D(p,I)$ and $x'\in[B(p,I)$ - $D(p,I)$]), it is clear that NGSE will be violated.

Since $x,x'\in B(p',I')$, and $x'\in[B(p,I)$ - $D(p,I)$], noting IE, we must have $[p'.x'=I'\ge p'.x]$ and $[p.x=I\ge p.x']$. Consider therefore the remaining case: $[p'.x'=I'>p'.x]$ and $[p.x=I\ge p.x']$. If $[p'.x'=I'>p'.x]$ and $[p.x=I=p.x']$, then, NGSE must be violated.

Therefore, we only need to rule out the remaining case where the following conditions both hold :

$$p'.x'=I'>p'.x\,; \tag{6}$$

and

$$p.x = I > p.x' . \qquad (7)$$

Consider $t \in (0,1)$, such that $[tp + (1-t)p'].x = [tp + (1-t)p'].x'$ (given (6) and (7), such t exists). Let $p^* = tp + (1-t)p'$ and let $I^* = p^*.x = p^*.x'$. We then have, noting IE, for all $x^* \in D(p^*, I^*)$:

$$tI + (1-t)I' > tp.x + (1-t)p'.x = I^* = p^*.x^* = tp.x^* + (1-t)p'.x^* .$$

Hence, either $p.x^* < I$ or $p'.x^* < I'$. Without loss of generality, suppose $p.x^* < I$. Then, noting IE, we have: $[p^*.x^* = p^*.x]$ and $[p.x^* < p.x]$, which imply $(p^* - p)(x^* - x) > 0$. This violates NGSE. •

Proof of Corollary 3:

Let the demand correspondence D satisfy IE and WARP. Then, by Proposition 1, D satisfies NGSE. We show that IE and NGSE imply homogeneity of degree 0.

Let $(p, I), (p', I') \in Q$ be such that, for some positive number λ, $p' = \lambda p$ and $I' = \lambda p$. Suppose $D(p, I) \neq D(p', I')$. Without loss of generality, assume that $x' \in D(p', I')$ but $x' \notin D(p, I)$. Let $x \in D(p, I)$. Since $B(p, I) = B(p', I')$, $x' \notin D(p, I)$, $x' \in D(p', I')$, $x \in D(p, I)$, and IE holds, we have $x' \in B(p, I) - D(p, I)$; $p'.x' = I'$, and, hence, $p.x' = I$; and $p.x = I$, and, hence, $p'.x = I'$. Noting $x \in D(p, I)$, $x' \in B(p, I) - D(p, I)$, $p'.x = I'$, and $x' \in D(p', I')$, by NGSE we have $(p - p').(x - x') < 0$. At the same time, noting $p'.x = p'x' = I'$ and $p.x = p.x' = I$, we have $(p - p').(x - x') = 0$. Thus we have a contradiction. •

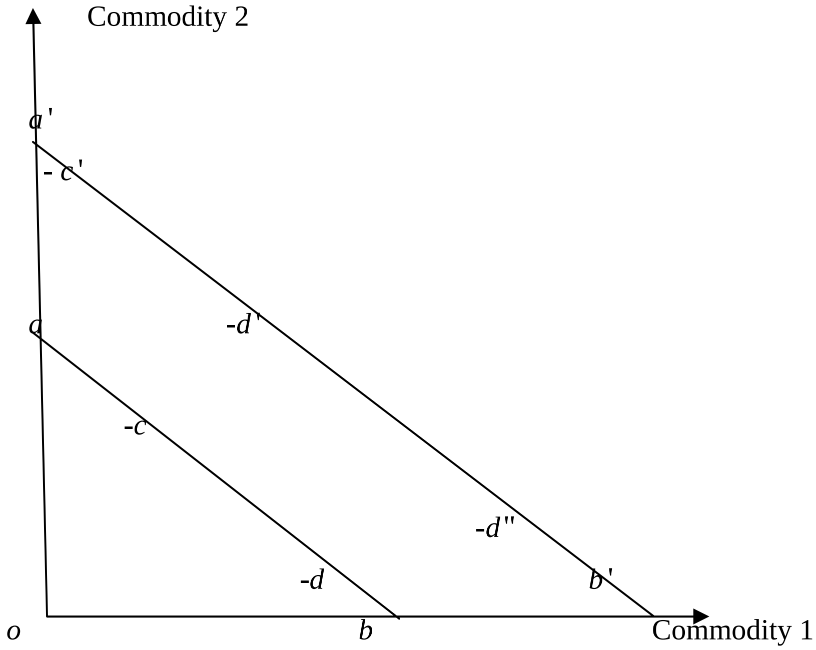

Figure 1