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ABSTRACT

Delinquent Networks*

Delinquents are embedded in a network of relationships. Social ties among delinquents are modeled by means of a graph where delinquents compete for a booty and benefit from local interactions with their neighbors. Each delinquent decides in a non-cooperative way how much delinquency effort he will exert. Using the network model developed by Ballester et al. (2006), we characterize the Nash equilibrium and derive an optimal enforcement policy, called the key-player policy, which targets the delinquent who, once removed, leads to the highest aggregate delinquency reduction. We then extend our characterization of optimal single player network removal for delinquency reduction, the key player, to optimal group removal, the key group. We also characterize and derive a policy that targets links rather than players. Finally, we endogenize the network connecting delinquents by allowing players to join the labor market instead of committing delinquent offenses. The key-player policy turns out to be much more complex since it depends on wages and on the structure of the network.

JEL Classification: A14, C72, K42, L14

Keywords: social networks, delinquency decision, key group, NP-hard problem, crime policies

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1 Introduction

Polls show that people regard crime and delinquency as the number one social problem. As such, identifying the root causes of delinquent activity and designing efficient policies against delinquency are two natural scopes for the economics profession. About thirty years ago, the major breakthrough in the economic analysis of crime was the work of Becker (1968) in which delinquents are rational individuals acting in their own self-interest. In deciding to commit a delinquency, delinquents weigh the expected costs against the expected benefits accruing from this activity. The goal of the criminal justice system is to raise expected costs of delinquency to offenders above the expected benefits. People will commit crimes only so long as they are willing to pay the prices society charges.

There is by now a large literature on the economics of crime. Both theoretical and empirical approaches have been developed over the years in order to better understand the costs and benefits of crime (see, for instance, the literature surveys by Garoupa, 1997, and Polinsky and Shavell, 2000). In particular, the interaction between the "delinquency market" and the other markets has important general equilibrium effects that are crucial if one wants to implement the most effective policies.[1] The standard policy tool to reduce aggregate delinquency that is common to all these models relies on the deterrence effects of punishment, i.e., the planner should increase uniformly punishment costs.

It is however well-established that delinquency is, to some extent, a group phenomenon, and the source of crime and delinquency is located in the intimate social networks of individuals (see e.g. Sutherland, 1947, Sarnecki, 2001 and Warr, 2002). Indeed, delinquents often have friends who have themselves committed several offences, and social ties among delinquents are seen as a means whereby individuals exert an influence over one another to commit crimes. In fact, not only friends but also the *structure* of social networks matters in explaining individual's own delinquent behavior. In adolescents' friendship networks, Haynie (2001) and Calvó-Armengol *et al.* (2005, 2009) show that individual Bonacich centrality (a standard measure of network centrality) together with the density of friendship links condition the delinquency-peer association. This suggests that the underlying structural properties of friendship networks must be taken into account to better understand the impact of peer influence on delinquent behavior and to address adequate and novel delinquency-reducing policies.

In this paper, we develop an explicit delinquent network game where individuals decide non-cooperatively their crime effort by using the network model developed by Ballester *et al.* (2006) to the case of delinquent networks. For this purpose, we build on the Beckerian incentives approach

[1] For example, Burdett *et al.* (2003) and Huang *et al.* (2004) study the interaction between crime and unemployment, while Verdier and Zenou (2004) analyze the impact of the land market on criminal activities. Others have focused on the education market (Lochner, 2004) or on political economy aspects of crime (İmrohoroğlu *et al.*, 2000). Most of these models generate multiple equilibria that can explain why identical areas may end up with different amounts of crime.

to delinquency behavior but let the cost to commit delinquent offenses to be determined, in part, by one's network of delinquent mates. We then consider different policies that aim at reducing the total crime activity in a delinquent network. Compared to Ballester *et al.* (2006), the present paper has the following innovations: (*i*) the payoff function contains a component with global strategic substitutes and is parameterized so that the effects of different parameters can be easily interpreted; (*ii*) it compares the effects of an increase in "punishment" and other standard crime policies with a "key player" policy; (*iii*) it analyzes a "key group" and a "key link" policy, in addition to the "key player" policy; (*iv*) it shows that finding a "key group" is an NP-hard problem and provide a simple (greedy) algorithm and a bound for the degree of suboptimality of the algorithm's solution; (*v*) it characterizes the equilibrium of a so-called "entry game" where individuals decide whether to continue participating in a delinquent network (in their previously allotted position) or take some job in the outside world; (*vi*) it analyzes the "key player" and "key group" policies in the "entry game." To the best of our knowledge, this is the first paper analyzing policies aiming at reducing crime in an explicit social network framework.[2]

Let us now be more precise about what we do in this paper. Following Ballester *et al.* (2006), we develop a delinquent network model where the payoff interdependence is, at least in part, rooted in the network links across players (see, in particular, the recent literature surveys by Goyal, 2007 and Jackson, 2008). At the Nash equilibrium, we obtain a relationship between equilibrium strategic behavior and network topology, as captured by the *Katz-Bonacich centrality measure*. This measure is an index of *connectivity* that not only takes into account the number of direct links a given delinquent has but also all his indirect connections.[3] In our delinquency game, the network payoff interdependence is restricted to direct network mates. But, because clusters of direct friends overlap, this local payoff interdependence spreads all over the network. At equilibrium, individual decisions emanate from all the existing network chains of direct and indirect contacts stemming from each player, a feature characteristic of Katz-Bonacich centrality.

Because network chains of contacts often overlap, the values of individual centrality indices are interrelated, which further translates into the interdependence of individual delinquency outcomes, and between individual and group (average) outcomes. This dependence of individual on group behavior is usually referred to as *peer effects* in the literature.[4] Peer effects are an intragroup externality, *homogeneous* across group members, that captures the *average* influence that members exert on each other. In our model, though, the peer effect influence varies across delinquents with their equilibrium-Bonacich centrality measure. The intragroup externality we obtain is *heteroge-*

[2] Calvó-Armengol and Zenou (2004) proposed a network model of criminal activities but without looking at policy issues.

[3] There are, of course, other measures of centrality (for example the class of *betweenness* measures; see Wasserman and Faust, 1994).

[4] The empirical evidence collected so far in the economics literature suggests that peer effects are, indeed, quite strong in criminal decisions. See, for instance, Case and Katz (1991), Glaeser *et al.* (1996), Ludwig *et al.* (2001), Patacchini and Zenou (2006), Sirakaya (2006), Damm and Dustmann (2008), Bayer *et al.* (2009).

neous across delinquents, and this heterogeneity reflects asymmetries in network locations across group members.

The standard policy tool to reduce aggregate delinquency relies on the deterrence effects of punishment. By uniformly hardening the punishment costs borne by all delinquents, the distribution of delinquency efforts shifts to the left and the average (and aggregate) delinquency level decreases. This homogeneous policy tackles average behavior explicitly and does not discriminate among delinquents depending on their relative contribution to the aggregate delinquency level. Our previous results, though, associate a distribution of delinquency efforts to the network connecting them. In particular, the variance of delinquency efforts reflects the variance of network centralities. In this case, a targeted policy that discriminates among delinquents depending on their relative network location, and removes a few suitably selected targets from this network, alters the whole distribution of delinquency efforts, not just shifting it. In many cases, it may yield to a sharper reduction in aggregate delinquency than standard deterrence efforts. In practice, the planner may want to identify optimal network targets to concentrate (scarce) investigatory resources on some particular individuals, or to isolate them from the rest of the group, either through leniency programs, social assistance programs, or incarceration.

To characterize the network optimal targets, we use a new measure of network centrality, the *intercentrality measure*, proposed by Ballester *et al.* (2006). This measure solves the planner's problem that consists in finding and getting rid of the *key player*, i.e., the delinquent who, once removed, leads to the highest aggregate delinquency reduction. We show that the key player is, precisely, the individual with the highest intercentrality in the network.

At this point, it is important to note that, to implement the key-player policy, one does not need to have all the information about the exact structure of the network. Indeed, the planner does not need to know all the links each individual has but only needs to be able to rank delinquents according to their intercentrality measure.[5] This is less demanding in terms of information and it implies, in particular, that two different networks can lead to the same policy implication, i.e., the same key player to remove. Take for example a star-shaped network. Then it does not matter how many links has the central delinquent, or whether some peripheral delinquents have some direct link with each other, or even how large the network is. In all these cases, the planner will remove the central delinquent because this is the key player −the delinquent with the highest intercentrality measure. This is obviously an extreme case and in other networks one may need more information to identify the key player. But this simple example highlights the advantages of implementing a

[5] Note that an undirected unweighted network is fully characterized by $n(n-1)/2$ values −the list of actual network links. We show that two $n-$ dimensional vectors aggregate this information in an enough informative manner for our purposes: first, to identify crime behavior − equilibrium-Bonacich centrality− and second, to identify optimal policy targets −optimal inter-centality. We further show that the only valuable information to identify the optimal target provided by the vector of optimal-inter-centralities is of *ordinal* nature, which further reduces the informational requirements on the network structure to effectively implement this policy.

key-player policy.

We extend our characterization of optimal single player network removal for delinquency reduction, the *key player*, to optimal group removal, the *key group*. For this purpose, we generalize the intercentrality measure to groups of players. For a given group size, the key group is precisely the one with the highest value for such centrality measure among groups of exactly this size. Given that the individual intercentrality captures both direct and indirect effects on equilibrium Katz-Bonacich centrality measures, the generalization to a group of the intercentrality measure needs to account (once and only once) for all the cross-contributions that arise both within and outside the group. For this reason, and contrary to most centrality measures found in the literature, the group centrality index is not a straightforward aggregation of its members centrality indices.

We then consider an alternative policy which aims at optimally removing a link (or a set of links) between two individuals in order to minimize the total delinquency level. In some situations, the limitation of resources or the nature of the problem requires to choose optimally among the set of dependences among players. For instance, a social planner would like to optimally reduce the (communication) externalities among delinquents, subject to a restriction in the number of bilateral influences that can be targeted. This situation is interpreted as a problem of optimally removing a set of links from the network. We obtain a new centrality measure which is roughly proportional to the product of the Katz-Bonacich centrality indices of the two delinquents involved in the link.

Because the geometric intricacies of the delinquency network are explicitly taken into account in the characterization of optimal network targets, the implications of our policy prescriptions are quite different from the standard deterrence-based policies, where both the apprehension probability and punishment are increased uniformly. We show that the key player (group) policy displays amplifying effects, and the gains following the judicious choice of the key player (group) go beyond the differences in intercentrality measures between the selected targets and any other delinquents in the network. We also show that the relative gains from targeting the key player (group) instead of operating a selection at random of a delinquent in the delinquency network increase with the variability in intercentrality measures across delinquents. In other words, the key player (group) prescription is particularly well-suited for networks that display stark location asymmetries across nodes. Also, our policy prescriptions rely on centrality measures particularly robust to mispecifications in network data, and thus open the door to relatively accurate estimations of these measures with small samples of network data.

In the last part of the paper, we endogenize the network connecting delinquents by allowing players to join the labor market instead of committing delinquent offenses. The model is now richer since, apart from punishment, the outside wage is an additional delinquency-reducing policy tool available to the planner. We show that the key player policy prescription now depends both on network features and on the wage level.

The organization of the paper is as follows. In the next section, we expose the basic delinquency

network game, characterize the Nash equilibrium, prove its existence and uniqueness, and give the general comparative statics results. In section 3, we analyze the key-player, group-player and key-link policies. Section 4 is devoted to the endogenous participation of individuals to delinquent activities. We analyze, in particular, how the policies exposed in Section 3 are affected by this choice. Throughout the paper, we use the same example of a particular bridge network with 11 delinquents to illustrate all our results.

2 Delinquency network outcomes

2.1 The delinquency network game

The network[6] A *network* g is a set of ex-ante identical individuals $N = \{1, \ldots, n\}$ and a set of *links* between them. We assume $n \geq 2$. The $n-$square adjacency matrix $\mathbf{G}$ of a network g keeps track of the direct connections in this network. By definition, players i and j are directly connected in g if and only if $g_{ij} = 1$, (denoted by link ij), and $g_{ij} = 0$ otherwise.[7] Links need not be reciprocal, so that we may have $g_{12} = 1$ and $g_{21} = 0$. Only in some of our results we will explicitly impose this symmetry. By convention, $g_{ii} = 0$. Thus $\mathbf{G}$ is $(0,1)-$matrix with zeros on its diagonal.

The delinquency decision game We focus on petty crimes so we consider delinquents rather than criminals.[8] Consider some delinquency network g. Delinquents in the network decide how much effort to exert. We denote by x_i the delinquency effort level of delinquent i, and by $\mathbf{x} = (x_1, ..., x_n)$ the population delinquency profile.

Following Becker (1968), we assume that delinquents trade off the costs and benefits of delinquent activities to take their delinquency effort decision. The expected delinquency gains to delinquent i are given by:

$$u_i(\mathbf{x},g) = \underbrace{y_i(\mathbf{x})}_{proceeds} - \underbrace{p_i(\mathbf{x},g)}_{apprehension}\underbrace{f}_{fine} \tag{1}$$

The individual proceeds $y_i(\mathbf{x})$ correspond to the gross delinquency payoffs of delinquent i. Individual i gross payoff positively depends on i's delinquency involvement x_i, and on the whole population delinquency effort $\mathbf{x}$. The proceeds $y_i(\mathbf{x})$ indicate the *global*[9] payoff interdependence. The cost of committing delinquency $p_i(\mathbf{x}, g)f$ is also positively related to x_i as the apprehension probability increases with one's involvement in delinquency, hitherto, with one's exposure to deter-

[6] General definitions and notations of matrices and networks can be found in Appendix A.

[7] Our model can be extended to allow for weighted links in a straightforward way.

[8] It is well-documented that social interactions and peer effects are stronger for petty crimes than for other types of crimes (Glaeser *et al.*, 1996; Jacob and Lefgren, 2003; Patacchini and Zenou, 2008).

[9] That is, across all criminals in the network.

rence. In words, $p_i(\mathbf{x}, g)$ reflects *local* complementarities in delinquency efforts across delinquents directly connected through g.[10]

The crucial assumption made here is that delinquents improve illegal practice while interacting with their direct delinquent mates. In other words, we assume that the higher the criminal connections to a criminal and/or the higher the involvement in criminal activities of these connections, the lower his individual probability to be caught $p_i(\mathbf{x}, g)$. The idea is as follows. *There is no formal way of learning to become a criminal*, no proper "school" providing an organized transmission of the objective skills needed to undertake successful criminal activities. Given this lack of formal institutional arrangement, we believe that the most natural and efficient way to learn to become a criminal is through the interaction with other criminals. Delinquents learn from other criminals belonging to the same network how to commit crime in a more efficient way by sharing the know-how about the "technology" of crime. In our model, we capture this local nature of the mechanism through which skills are acquired by relating the individual probability to be caught to the crime level involvement of one's direct mates, and by assuming that this probability decreases with the corresponding local aggregate level of crime.

This view of criminal networks and the role of peers in learning the technology of crime is not new, at least in the criminology literature. In his very influential theory of differential association, Sutherland (1947) locates the source of crime and delinquency in the intimate social networks of individuals. Emphasizing that criminal behavior is *learned* behavior, Sutherland (1947) argued that persons who are selectively or differentially exposed to delinquent associates are likely to acquire that trait as well.[11] In particular, one of his main propositions states that when criminal behavior is learned, the learning includes (i) techniques of committing the crime, which are sometimes very complicated, sometimes very simple, (ii) the specific direction of motives, drives, rationalization and attitudes. Interestingly, the positive correlation between self-reported delinquency and the number of delinquent friends reported by adolescents has proven to be among the strongest and one of the most consistently reported findings in the delinquency literature (for surveys, see War, 1996 and Matsueda and Anderson, 1998).

One natural way of interpreting the social connections between criminals is through a gang since the latter is in general viewed as a specific type of criminal network (Sarnecki, 2001). Indeed, when individuals belong to the same gang, they learn from each other. Using data from the Rochester Youth Development study, which followed 1,000 adolescents through their early adult years, Thornberry *et al.* (1993) find that once individuals become members of a gang, their rates

[10] See also Brock and Durlauf (2001) for a global/local dichotomy in capturing social interactions and Ioannides (2006) for an exhaustive analysis of the effects of network topology in the Brock and Durlauf setting. Observe that all our results remain unchanged if the local network externalities enter the benefit function instead of the cost function in (2) as long as network payoffs reflect net strategic substituability.

[11] Sutherland (1947) and Akers (1998) expressly argue that criminal behavior is learned from others in the same way that *all* human behavior is learned. Indeed, young people may be influenced by their peers in all categories of behavior - music, speech, dress, sports, and *delinquency*.

of delinquency increase substantially compared to their behavior before entering the gang. In other words, networks of criminals or gangs amplify delinquent behaviors. In the sociological literature, this is referred to as the *social facilitation* model, where gang members are intrinsically no different from nongang members in terms of delinquency or drug use. If they do join a gang, however, the normative structure and group processes of the gang (network) are likely to bring about high rates of delinquency and drug use. Gang membership is thus viewed as a major cause of deviant behavior. This is also what is found by Thornberry *et al.* (2003). In the present paper, the gang interpretation of the network is possible as long as it means that the role of gangs is to facilitate the learning of crime technology to its members without implying that crimes are committed collectively as it is sometimes the case in gang activities. In other words, in our model, individuals *learn* illegal conduct from others but *practice* it alone.

Since in most of the papers cited above (from the sociology and criminology literatures), selection and endogeneity issues are not properly addressed, we would like to provide some evidence on learning in crime from the economics literature where these econometric issues are taken into account. Damm and Dustmann (2008) investigate the following question: Does growing up in a neighborhood in which a relatively high share of youth has committed crime increase the individual's probability of committing crime later on? To answer this question, Damm and Dustmann exploit a Danish natural experiment that randomly allocates parents of young children to neighborhoods with different shares of youth criminals. With area fixed effects, their key results are that one standard deviation increase in the share of youth criminals in the municipality of initial assignment increases the probability of being charge with an offense at the age 18-21 by 8 percentages point (or 23 percent) for men. This neighborhood crime effect is mainly driven by property crime.[12] Bayer *et al.* (2009) consider the influence that juvenile offenders serving time in the same correctional facility have on each other's subsequent criminal behavior. They also find strong evidence of learning effects in criminal activities since exposure to peers with a history of committing a particular crime increases the probability that an individual who has already committed the same type of crime recidivates with that crime.[13]

There are clearly learning effects in crime. One may, however, argue that, although delinquents learn a lot from their best friends, the learning is not infinite so that after some time friends do not provide any positive externalities that may reduce the probability to be caught. This is clearly not true at least for delinquent friendships. Indeed, the techniques and information about crime are *not* static and constantly evolving. For example, friends may help a delinquent to be more efficient

[12] Without controlling for selection effects, Case and Katz (1991), using data from the 1989 NBER survey of youths living in low-income Boston neighborhoods, find that the direct effect of moving a youth with given family and personal characteristics to a neighborhood where 10 percent more of the youths are involved in crime than in his or her initial neighborhood is to raise the probabibility the youth will become involved in crime by 2.3 percent.

[13] Building on the binary choice model of Brock and Durlauf (2001), Sirakaya (2006) identifies social interactions as the primary source of recidivist behavior in the United States.

in shoplifting by explaining the new type of protection that has been installed in a particular shop. This information may not be valid a year later if the shop has changed its protection system. Friends can also tell a delinquent which apartment has already been robbed so that it avoid this delinquent to take risk without getting much proceeds. Another interesting example is people selling illegal DVDs on the street. They share knowledge based on experience (level of activity) but they decide on their effort separately. To summarize, delinquents who know each other can share information because they are friends (we take this communication as given) even though they act separately in the delinquent world.

For the sake of tractability, we restrict to the following simple expressions:

$$\begin{cases} y_i(\mathbf{x}) = x_i \max\left\{A - \delta \sum_{j=1}^n x_j, 0\right\} \\ p_i(\mathbf{x},g) = p_0 x_i \max\left\{1 - \phi \sum_{j=1}^n g_{ij} x_j, 0\right\} \end{cases} \tag{2}$$

where $A > 0$, $\delta > 0$ and $\phi \geq 0$. For the sake of simplicity, we take $A = 1$. We assume that, at an equilibrium $\mathbf{x}^*$:[14]

$$1 - \delta \sum_{j=1}^n x_j^* \geq 0 \text{ and } 1 - \phi \sum_{j=1}^n g_{ij} x_j^* \geq 0$$

so that, by direct substitution, we can focus on the following utility function:

$$u_i(\mathbf{x},g) = (1-\pi) x_i - \delta x_i^2 - \delta \sum_{j \neq i}^n x_i x_j + \pi\phi \sum_{j=1}^n g_{ij} x_i x_j \tag{3}$$

where $\pi = p_0 f$ is the the marginal expected punishment cost for an isolated delinquent. We assume throughout that $\pi < 1$. With these expression, we have:

$$\sigma_{ij} = \frac{\partial^2 u_i(\mathbf{x},g)}{\partial x_i \partial x_j} = \begin{cases} \overline{\sigma} = -\delta + \pi\phi g_{ij} & \text{if} \quad g_{ij} = 1 \\ \underline{\sigma} = -\delta & \text{if} \quad g_{ij} = 0 \end{cases} \tag{4}$$

so that $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$, for all $i \neq j$ with $\underline{\sigma} \leq 0$. The parameter $\delta \geq 0$ measures the intensity of the *global* interdependence on gross delinquency payoffs. Here, individual delinquency efforts are global strategic substitutes. The optimal delinquency effort of a given delinquent thus decreases with the delinquency involvement of any other delinquent in the network. The expression $\pi\phi > 0$ captures the *local* strategic complementarity of efforts on the apprehension probability.[15] This expression is non-zero only when $g_{ij} = 1$, that is, when delinquents i and j are directly linked to each other. Finally, note that $\partial^2 u_i(\mathbf{x},g)/\partial x_i^2 = -2\delta < 0$.

[14] This assumption is satisfied, for instance, when $\delta \geq \phi$ or $\pi \geq 1/2$.

[15] A different scenario arises if we assume that players face substitutability in actions at the local level ($\phi < 0$), which results in a game with substitutabilities. Bramoullé *et al.* (2008) provide an exhaustive theoretical analysis of this case.

2.2 The Katz-Bonacich network centrality measure

Let $\mathbf{G}^k$ be the kth power of $\mathbf{G}$, with coefficients $g_{ij}^{[k]}$, where k is some integer. The matrix $\mathbf{G}^k$ keeps track of the indirect connections in the network: $g_{ij}^{[k]} \geq 0$ measures the number of walks[16] of length $k \geq 1$ in g from i to j. In particular, $\mathbf{G}^0 = \mathbf{I}$.

Given a scalar $a \geq 0$ and a network g, we define the following matrix:

$$\mathbf{M}(g,a) = [\mathbf{I} - a\mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k.$$

These expressions are all well-defined for low enough values of a.[17] The parameter a is a decay factor that scales down the relative weight of longer walks.

If $\mathbf{M}(g,a)$ is a non-negative matrix, its coefficients $m_{ij}(g,a) = \sum_{k=0}^{+\infty} a^k g_{ij}^{[k]}$ count the number of walks in g starting from i and ending at j, where walks of length k are weighted by a^k.

Let $\mathbf{1}$ be the $n-$dimensional vector of ones.

Definition 1 *Consider a network g with adjacency $n-$square matrix $\mathbf{G}$ and a scalar a such that $\mathbf{M}(g,a) = [\mathbf{I} - a\mathbf{G}]^{-1}$ is well-defined and non-negative. The vector of Katz-Bonacich centralities of parameter a in g is:*

$$\mathbf{b}(g,a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot \mathbf{1} \tag{5}$$

The Katz-Bonacich centrality[18] of node i is $b_i(g,a) = \sum_{j=1}^{n} m_{ij}(g,a)$, and counts the *total* number of walks in g starting from i. By definition, $m_{ii}(g,a) \geq 1$, and thus $b_i(g,a) \geq 1$, with equality when $a = 0$.

2.3 Nash equilibrium

For all $\mathbf{y} \in \mathbb{R}^n$, $y = y_1 + ... + y_n$ is the sum of its coordinates. Define $b(g,\theta) = \sum_{i=1}^{i=n} b_i(g,\theta)$, denote $\theta = \pi\phi/\delta$ and let $\rho(g)$ be the *spectral radius* of the adjacency matrix $\mathbf{G}$ (see Appendix A). We have the following result:[19]

Proposition 1 *If $\theta\rho(g) < 1$, then there exists a unique Nash equilibrium $\mathbf{x}^*$, which is interior, and given by:*

$$\mathbf{x}^* = \frac{1-\pi}{\delta\left[1 + b(g,\theta)\right]}\mathbf{b}(g,\theta) \tag{6}$$

The equilibrium Katz-Bonacich centrality measure $\mathbf{b}(g,\theta)$ is thus the relevant network characteristic that shapes equilibrium behavior. The condition $\theta\rho(g) < 1$ relates the payoff function

[16] See Appendix A for the definition of a walk.

[17] Take a smaller than the spectral radius of $\mathbf{G}$, defined in Appendix A.

[18] due to Katz (1953) and Bonacich (1987).

[19] All proofs are given in Appendix B.

to the network topology. When this condition holds,[20] the ratio of the local to the global payoff interdependence $\theta = \pi\phi/\delta$ is lower than the inverse of the spectral radius of the adjacency matrix $\mathbf{G}$ of the network g, which is a measure of connectivity in the network g. Let $l(g) \equiv \mathbf{1}^\top \mathbf{G}\mathbf{1}$, that, is, the number of links in the network g. As a matter of fact, $\rho(g) \leq \sqrt{l(g) + n - 1}$, so that $\theta\sqrt{l(g) + n - 1} < 1$ is a stronger sufficient condition that dispenses from computing the spectral radius of $\mathbf{G}$. In this case (and only then), the matrix $[\mathbf{I} - \theta\,\mathbf{G}]^{-1}$ can be developed into the infinite sum $\sum_{k\geq 0} \theta^k \mathbf{G}^k$, which brings the Katz-Bonacich centrality measure into the picture.

The game we analyze here belongs to games with *complementarities*, for which the cross-payoffs derivatives between every pair of players are non-negative. The condition $\theta\rho(g) < 1$ guarantees that local complementarities are not too large compared to own concavity. When this condition does not hold, existence of equilibrium becomes an issue because the strategy space is unbounded. The literature on supermodular games (see Topkis, 1979, Amir, 2005, Vives, 2005, for literature surveys) has dealt with this problem by imposing a bounded lattice on the strategy space.

Consistent with the predictions of our model, two recent empirical studies by Haynie (2001) and Calvó-Armengol *et al.* (2005) show that structural properties of friendship networks indeed condition the association between friends' delinquency and an individual's own delinquent behavior.[21] Also, by analyzing the network organization of conspiracy, Baker and Faulkner (1993) show that a measure of network centrality based on direct links predicts the individual probability to be apprehended and convicted as well as the magnitude of the fine.

2.4 Comparative statics

In Proposition 1, the individual and aggregate delinquency levels depend on the underlying network g connecting them through the adjacency matrix $\mathbf{G}$ in (6). The next result establishes a positive relationship between the equilibrium aggregate delinquency level and the network pattern of connections.

We write $g \subset g'$ to denote that the set of links in g' contains the links in g, i.e.,for all i, j, $g'_{ij} = 1$ if $g_{ij} = 1$.

Proposition 2 *Let g and g' be symmetric networks such that $g \subset g'$. If $\theta\rho(g') < 1$, then, in equilibrium, the total delinquency level under g' is strictly higher than that under g.*

Consider two nested networks g and g' such that $g \subset g'$. Then, either g and g' connect the same number of delinquents but there are more direct links between them in g' than in g, or g' brings additional individuals into the pool of delinquents already connected by g, or both. Proposition

[20] Testing the impact of the Katz-Bonacich centrality measure on educational and criminal outcomes in the United States, Calvó-Armengol *et al.* (2005, 2009) found that only 18 out of 199 networks (i.e. 9 percent) do not satisfy this eigenvalue condition.

[21] Both studies use data from the National Longitudinal Study of Adolescent Health, United States, 1994-1995.

2 shows that the density of network links and the network size (or boundaries) affect positively aggregate delinquency, a feature often referred to as the *social multiplier effect*.[22]

3 Delinquency network policies

3.1 Finding the key player

The standard policy tool to reduce aggregate delinquency relies on the deterrence effects of punishment (see for example Becker, 1968). Formally, an increase in π, which translates into an increase in θ, amounts to hardening punishment costs borne by delinquents. Our previous results associate a distribution of delinquency efforts across delinquents to any delinquency network connecting them. In this case, an increase in θ affects all delinquency decisions simultaneously and shifts the whole delinquency efforts distribution to the left, thus reducing the average (and the aggregate) delinquency level.

In our model, though, delinquent behavior is tightly rooted in the network structure. When all delinquents hold homogeneous positions in the delinquency network, they all exert a similar delinquency effort. In this case, the above-mentioned policy, that tackles average behavior and does not discriminate among delinquents depending on their relative contribution to the aggregate delinquency level, may be appropriate. However, if delinquents hold very heterogeneous positions in the delinquency network, they contribute very differently to the aggregate delinquency level. The variance of efforts is higher. In this case, we could expect a sharp reduction in average delinquency by directly removing a delinquent from the network and thus altering the whole distribution of delinquency efforts, not just shifting it. A targeted policy that discriminates among delinquents depending on their location in the network may then be more appropriate.

The key player is the one inducing the highest aggregate delinquency reduction. Given that delinquent removal has both a direct and an indirect effect on the group outcome, the choice of the key player results from a compromise between both effects. In particular, the key player need not necessarily be the one exerting the highest delinquency effort or, equivalently, the one with the highest centrality measure. The planner's objective is thus to generate the highest possible reduction in aggregate delinquency level by picking the appropriate delinquent. Formally, the planner's problem is the following:

$$\max\{x^*(g) - x^*(g_{-i}) \mid i = 1, ..., n\},$$

which, when the original delinquency network g is fixed, is equivalent to:

$$\min\{x^*(g_{-i}) \mid i = 1, ..., n\} \tag{7}$$

From Ballester *et al.* (2006), we now define a new network centrality measure $d(g, \theta)$ that will happen to solve this compromise.

[22] See, for instance, Glaeser *et al.* (2003), and references therein.

Definition 2 *For all network g and for all i, the measure*

$$\begin{aligned} d_i(g,\theta) &= b(g,\theta) - b(g_{-i},\theta) \\ &= \frac{b_i(g,\theta)^2}{m_{ii}(g,\theta)} \end{aligned} \tag{8}$$

accounts for the number of walks that crosses player i in the network i.

The intercentrality measure $d_i(g,\theta)$ of delinquent i is the sum of i's centrality measures in g, and i's contribution to the centrality measure of every other delinquent $j \neq i$ also in g. It accounts both for one's exposure to the rest of the group and for one's contribution to every other exposure.

The following result establishes that intercentrality captures, in an meaningful way, the two dimensions of the removal of a delinquent from a network, namely, the direct effect on delinquency and the indirect effect on others' delinquency involvement.

Proposition 3 *A player i^* is the key player that solves (7) if and only if i^* is a delinquent with the highest intercentrality in g, that is, $d_{i^*}(g,\theta) \geq d_i(g,\theta)$, for all $i = 1, ..., n$.*

Observe that the key player policy is such the planner perturbs the network by removing a delinquent and all other delinquents are allowed to change their effort after the removal but the network is not "rewired", i.e. individuals do not optimally change their relationships (links) with their friends. This assumption can be justified for two reasons. First, it would be extremely difficult to solve a network formation problem every time a player is removed. Second, in the context of a *short-term policy* and because friendship relationships take longer to adjust than the level of criminal activity, it is reasonable to assume that delinquents do not change their friends when one of them is removed even though they can modify their crime activity.

Example Consider the network g in Figure 1 with eleven delinquents.

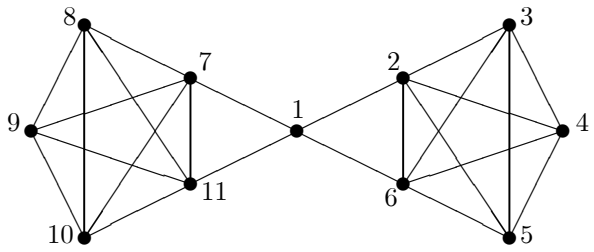


Figure 1

We distinguish three different types of equivalent actors in this network, which are the following:

Type	Players
1	1
2	2, 6, 7 and 11
3	3, 4, 5, 8, 9 and 10

From a macro-structural perspective, type−1 and type−3 delinquents are identical: they all have four direct links, while type −2 delinquents have five direct links each. From a micro-structural perspective, though, delinquent 1 plays a critical role by bridging together two closed-knit (fully intraconnected) communities of five delinquents each. By removing delinquent 1, the network is maximally disrupted as these two communities become totally disconnected, while by removing any of the type−2 delinquents, the resulting network has the lowest aggregate number of network links.

We identify the key player in this network of delinquents. If the choice of the key player were solely governed by the *direct* effect of delinquent removal on aggregate delinquency, type−2 delinquents would be the natural candidates. Indeed, these are the ones with the highest number of direct connections. But the choice of the key player needs also to take into account the *indirect* effect on aggregate delinquency reduction induced by the network restructuring that follows the removal of one delinquent from the original network. Because of his communities' bridging role, delinquent 1 is also a possible candidate for the preferred policy target.

Table 1 computes, for delinquents of types 1, 2 and 3, the value of delinquency efforts x_i, centrality measures $b_i(g,\theta)$ and intercentrality measures $d_i(g,\theta)$ for different values of θ and with $\delta = \phi = 1$. In each column, a variable with a star identifies the highest value.[23]

θ	0.1			0.2		
Player Type	x_i	b_i	d_i	x_i	b_i	d_i
1	0.077	1.75	2.92	0.072	8.33	41.67*
2	0.082*	1.88*	3.28*	0.079*	9.17*	40.33
3	0.075	1.72	2.79	0.067	7.78	32.67

First note that type−2 delinquents always display the highest $b-$centrality measure. These delinquents have the highest number of direct connections. Besides, they are directly connected to the bridge delinquent 1, which gives them access to a very wide and diversified span of indirect connections. Altogether, they are the most $b-$central delinquents.

For low values of θ, the direct effect on delinquency reduction prevails, and type−2 delinquents are the key players −those with highest intercentrality measure d_i. When θ is higher, though, the most active delinquents are not anymore the key players. Now, indirect effects matter a lot, and eliminating delinquent 1 has the highest joint direct and indirect effect on aggregate delinquency reduction.

When the punishment cost θ is low, delinquents transfer their know-how only at a very local level, with their direct delinquent mates. When θ increases, delinquents counter the higher deterrence they face by spreading their know-how further away in the network and establishing synergies with delinquents located in distant parts of the social setting. In this latter case, the optimal tar-

[23] We can compute the highest possible value for θ compatible with our definition of centrality measures, equals to $\widehat{\theta} = \frac{2}{3+\sqrt{41}} \simeq 0,213$.

geted policy is the one that maximally disrupts the delinquency network, thus harming the most its know-how transferring ability.

Note that the network g_{-1} has twenty different links, while g_{-2} has nineteen links. In fact, when θ is small enough, the key player problem minimizes the number of remaining links in a network, which explains why type-2 delinquents are the key player when $\theta = 0.1$ in this example.

The individual Nash equilibrium efforts of the delinquency-network game are proportional to the *equilibrium Katz-Bonacich centrality* network measures, while the key player is the delinquent with the highest *intercentrality measure*. As the previous example illustrates, these two measures need not to coincide. This is not surprising, as both measures differ substantially in their foundation. Whereas the equilibrium-Katz-Bonacich centrality index derives from strategic individual considerations, the intercentrality measure solves the planner's optimality collective concerns. In particular, the equilibrium Katz-Bonacich centrality measure fails to internalize all the network payoff externalities delinquents exert on each other, while the intercentrality measure internalizes them all. More formally, the measure $d(g, \theta)$ goes beyond the measure $\mathbf{b}(g, \theta)$ by keeping track of all the cross-contributions that arise between its coordinates $b_1(g, \theta), ..., b_n(g, \theta)$.

Definition 2 specifies a clear relationship between $d(g, \theta)$ and $\mathbf{b}(g, \theta)$. Holding $b_i(g, \theta)$ fixed, the intercentrality $d_i(g, \theta)$ of player i decreases with the proportion $m_{ii}(g, \theta)/b_i(g, \theta)$ of i's Katz-Bonacich centrality due to self-loops, and increases with the fraction of i's centrality amenable to out-walks.

3.2 Comparing policies

The cost of finding the key player Given a delinquency network g and a punishment cost θ, the ranking of delinquents according to their individual intercentrality measure $d_i(g, \theta)$s provides a criterion for the selection of an optimal target in the network. Implementing such a network-based policy has obviously its costs. Indeed, the computation of the intercentrality measures relies on the knowledge of the adjacency matrix of the delinquency network. This matrix is obtained from sociometric data that identifies the network links between delinquents. It is important to note that sociometric data on delinquency is available in many cases. For instance, Haynie (2001) and Calvó-Armengol *et al.* (2005) use friendship data to identify delinquent peer networks for adolescents in 134 schools in the U.S. that participated in an in-school survey in the 1990's. Sarnecki (2001) provides a comprehensive study of co-offending relations and corresponding network structure for football hooligans and right-wing extremists in Stockholm. Baker and Faulkner (1993) reconstruct the structure of conspiracy networks for three well-known cases of collusion in the heavy electrical equipment industry in the U.S. In all these cases, one may directly use the available data to compute the intercentrality measures.

In some other cases, though, *ad hoc* information gathering programs have to be implemented. Interestingly, Costebander and Valente (2003) show that centrality measures based on connectivity

(rather than betweenness), such as **b** and d, are robust to mispecifications in sociometric data, and thus open the door to estimations of centrality measures with incomplete samples of network data. This, obviously, reduces the cost of identifying the key player. The idea behind these results is that these measures take into account *all* walks in the network. Thus, generally the centrality of a player is not determined only by his direct links but by the complete structure of the network. In this sense, the probability that a missing link affects the choice of the most central/intercentral player is smaller than with other type of measures. This difference turns significant the higher the value of the density parameter θ since, in that case, higher order walks are also taken into account in computing the centrality/intercentrality of a player.

Key player versus random target To fully assess the relevance of the key player delinquency policy, we also need to evaluate the relative returns from following this network targeted policy. For this purpose, we compare the reduction in aggregate delinquency following the elimination of the key player with respect to the expected consequences when the target is selected randomly.

For each delinquent i in the delinquency network, define:

$$\eta_i(g) = n\frac{x^*(g) - x^*(g_{-i})}{\sum_{j=1}^{n}[x^*(g) - x^*(g_{-j})]}.$$

This is the ratio of returns (in delinquency reduction) when i is the selected target versus a random selection with uniform probability for all delinquents in the network.

Denote by $\overline{d}(g,\theta)$ the average of the intercentrality measures in network g, and by $\sigma_{\mathbf{d}}(g,\theta)$ the standard deviation of the distribution of this intercentrality measures. The following result establishes a lower bound on the ratio of returns in delinquency reduction when the key player is removed.

Proposition 4 *Let i^* be the key player in g for a given θ. Then,*

$$\eta_{i^*}(g) \geq 1 + \frac{\sigma_{\mathbf{d}}(g,\theta)}{\overline{d}(g,\theta)}.$$

The relative gains from targeting the key player instead of operating a selection at random in the delinquency network increase with the variability in intercentrality measures across delinquents as captured by $\sigma_{\mathbf{d}}(g,\theta)$. In other words, the key player prescription is particularly well-suited for networks that display stark location asymmetries across nodes. In these cases, it is more likely than the relative gains from implementing such a policy compensate for its relative costs.

Key player versus standard deterrence policy Consider the key player removal policy. When a delinquent is removed from the network, the intercentrality measures of all the delinquents that remain active are reduced, that is, $d_j(g_{-i^*},\theta) \leq d_j(g,\theta)$, for all $j \neq i^*$, which triggers a

decrease in delinquency involvement for all of them. Moreover, when delinquent i^* is removed from the delinquency network, the corresponding ratio of aggregate delinquency reduction with respect to the network centrality reduction is an increasing function of the intercentrality measure $d_i(g,\theta)$ of this delinquent. Formally,

$$\frac{\partial}{\partial d_i(g,\theta)}\left[\frac{x^*(g)-x^*(g_{-i})}{b(g,\theta)-b(g_{-i},\theta)}\right]>0.$$

In words, *the target policy displays amplifying effects*, and the gains following the judicious choice of the key player (the one with highest intercentrality measure) go beyond the differences in intercentrality measures between this player and any other delinquent in the network.

Consider standard deterrence ("uniform") policies that consist in increases in θ. In particular, consider policies increasing π (i.e. increase in the fine f), or reducing δ, or increasing ϕ.

Observe first that an increase of π above 1 would induce an equilibrium with no delinquency. The problem is that the condition $\frac{\pi\phi}{\delta}\rho(g)<1$ in Proposition 1 that guarantees the existence of a unique interior Nash equilibrium and that the Bonacich centrality measure is well-defined may not be anymore satisfied. Moreover, we are interested in situations where it is *costly* for the authorities to increase π (or to implement any other policy). A thorough analysis of how the costs of different policies affect the choice of the "right" policy is, however, beyond the scope of this paper.

Let us now focus on the effect of π, δ, or ϕ on $x^*=\mathbf{1}^T.\mathbf{x}^*$, the equilibrium aggregate delinquency activity. Observe that we are dealing with the situation of a unique equilibrium under Proposition 1. From expression (6), it is easy to obtain:

$$x^*=\frac{(1-\pi)\,b(g,\theta)}{\delta\left[1+b(g,\theta)\right]} \tag{9}$$

It is then straightforward to show that the aggregate delinquency activity x^* is increasing in the local complementarity parameter ϕ but is decreasing in the global substitutability parameter δ. However, the effect of π on x^* is ambiguous and given by

$$\frac{\partial x^*}{\partial \pi}=\underbrace{-\frac{x^*}{1-\pi}}_{\text{direct negative effect}}+\underbrace{\frac{\phi}{\delta}\frac{\partial x^*}{\partial \theta}}_{\text{indirect positive effect}}$$

where

$$\frac{\partial x^*}{\partial \theta}=\frac{(1-\pi)}{\delta}\frac{1}{\left[1+b(g,\theta)\right]^2}\frac{\partial b(g,\theta)}{\partial \theta}>0$$

The impact on punishment results from the combination of two effects that work in opposite directions. First, the individual probability to be apprehended, and thus the punishment costs borne by each delinquent, increase with π. This is a *direct negative effect*. Second, when π increases, delinquents react strategically by acquiring a better delinquency technology to thwart the higher deterrence they now face. The improvement in delinquency technology stems from more intense

know-how inflows and transfers in the delinquency network. Each delinquent centrality measure $b_i(g, \theta)$ increases, which translates into a higher delinquency involvement for each delinquent. This is an *indirect positive effect* on aggregate delinquency that mitigates the direct negative effect. In order to better understand this last effect, we run numerical simulations for $\delta = 0.1$ and for which the maximum value of π is consistent with the spectral condition of Proposition 1. The results are given in Figures 2a ($\phi = 0.8$) and 2b ($\phi = 0.1$).

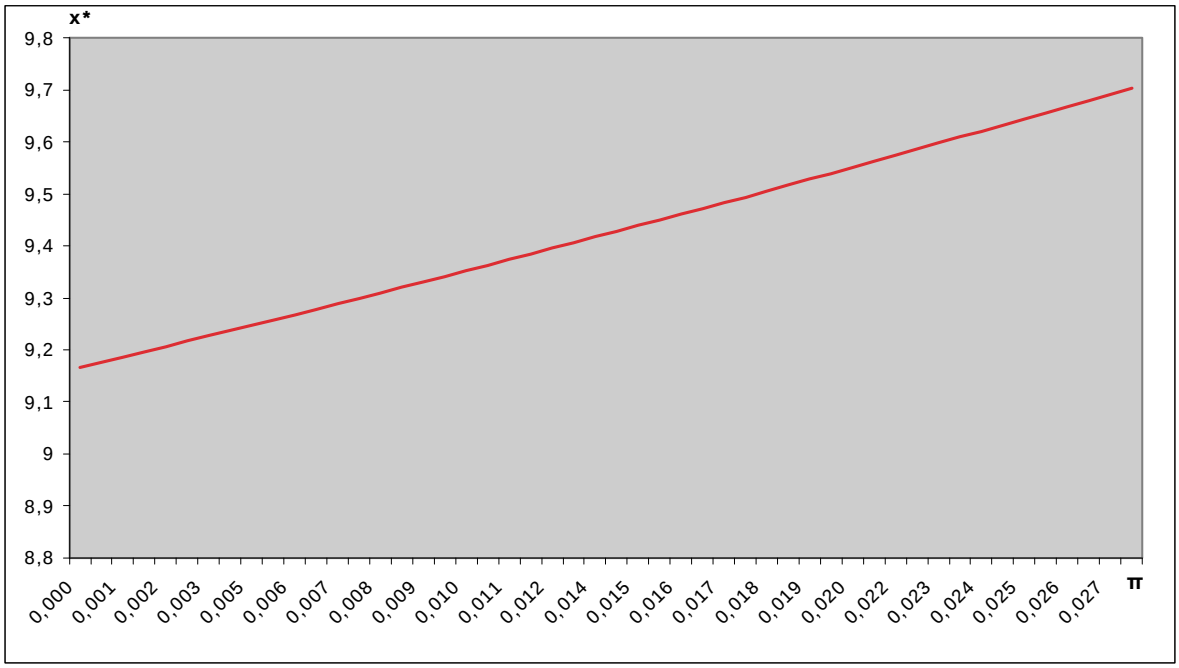


Figure 2a. Impact of deterrence π on total level of delinquent activity for $\delta = 0.1$ and $\phi = 0.8$.

We observe how the actual density of the network ϕ determines the sign of this effect. In Figure 2a, the network is more connected ($\phi = 0.8$) and there are a lot of synergies between delinquents. Bonacich centralities have high values, meaning that both direct and direct links matter very much. In that case, increasing punishment π *increases* total delinquent activity because the indirect positive effect dominates the direct negative effect. In Figure 2b, interactions are not important between criminals since $\phi = 0.1$. This means in particular, that friends of friends have not that much influence on delinquents. As a result, the network effect becomes unimportant compared to the deterrence effect and an increase in punishment π reduces total delinquency x^*. These results imply that the policy maker should be aware of the degree of connectivity of the network if it is to implement a deterrence policy aiming at reducing delinquent activity. In particular, if a network of delinquents is dense and well connected so that ϕ is high, it should be clear that a key-player policy will be more effective in reduce total delinquency than an increase in punishment.

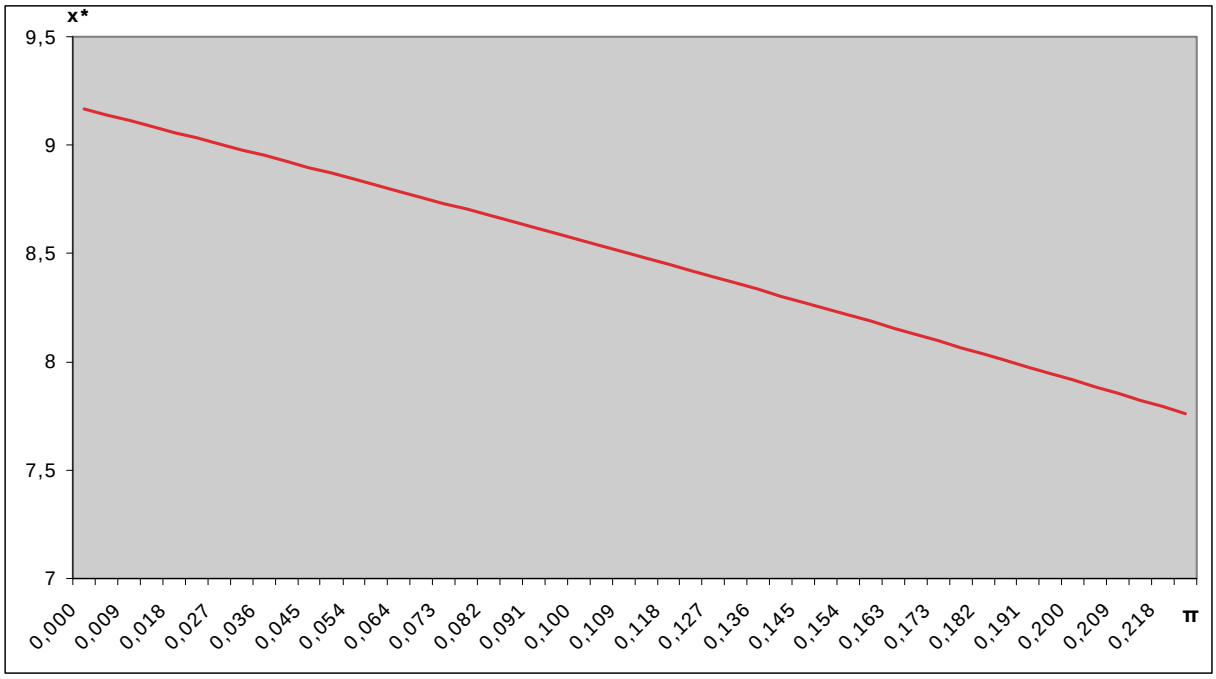


Figure 2b. Impact of deterrence π on total level of delinquent activity for $\delta = 0.1$ and $\phi = 0.1$.

3.3 From the key player to the key group

So far, we have characterized *optimal single player removal* from the network to reduce delinquency, a *key player*. We now characterize *optimal group removal* from the network, a *key group*.

3.3.1 Finding the key group of players

In our model, individual equilibrium behavior is tightly rooted in the network structure through (6). The removal of a set of players from the population, holding the pattern of social interactions among the other players fixed, alters the whole distribution of outcomes.

We will devote this section to identifying the optimal target set in the population when the planner wishes to reduce aggregate delinquency.[24]

We wish to eliminate a group of s players from the current population. If we remove a set S of players such that $|S| = s$, the network of delinquents becomes g_{-S}. The problem is therefore to minimize $x^*(g_{-S})$ by picking the adequate set S from the population. Formally, the planner

[24] Bollobás and Riordan (2003) contains a mathematical analysis of the relative network disruption effects of a topological attack versus random failures in large networks. See also Albert *et al.* (2000) for a numerical analysis for the case of the World Wide Web.

maximizes the total change in delinquent activity:

$$\max_{|S|\leq s} x^*(g) - x^*(g_{-S}),$$

which is equivalent to:

$$\min_{|S|\leq s} x^*(g_{-S}) \tag{10}$$

This is a finite optimization problem, which admits at least one solution. Let S^* be a solution to (10). We call the set S^* a *key group* of the game. Removing S^* from the game has the highest overall impact on delinquency.

In the following, we assume that the condition on eigenvalue in Proposition 1 holds in the game, guaranteeing the uniqueness of Katz-Bonacich solutions in any game induced by a subset of players. The reason is that, by Lemma 3 in Appendix A, $\rho(g) \geq \rho(g_{-S})$. As a consequence, if $\mathbf{b}(g,\theta)$ is well-defined and non-negative (as implied by the condition in Proposition 1), so is $\mathbf{b}(g_{-S},\theta)$.

Definition 3 *The group intercentrality of S in the network g is:*

$$d_S(g,\theta) = b(g,\theta) - b(g_{-S},\theta)$$

In fact, $d_S(g,\theta)$ is the weighted number of walks in g crossing some agent in S. The case $s = 1$ obviously corresponds to the case of finding the key player in the delinquency network.

As with the key player, it turns out that diminishing the aggregate delinquent activity reduces to choose the set with the highest group intercentrality:

$$\max_{|S|=s} d_S(g,\theta), \tag{11}$$

that is, the solution[25] of (7) is $S^* \subseteq N$ such that $d_{S^*}(g,\theta) \geq d_S(g,\theta)$, for all $S \subseteq N$ with $|S| = s$.

Remark 1 *An equivalent formulation of the key group problem (11) is:*

$$\max_{\{i_1,...,i_s\}\subseteq N} d_{i_1}(g,\theta) + d_{i_2}(g_{-\{i_1\}},\theta) + d_{i_3}(g_{-\{i_1,i_2\}},\theta) + ... + d_{i_s}(g_{-\{i_1,...,i_{s-1}\}},\theta), \tag{12}$$

where $i_1, ..., i_s$ are different two by two.

In words, the key group maximizes the sum of the individual intercentrality measures of its members across the networks obtained through sequential removal of these members.[26] The idea behind this expression is as follows. We must eliminate a set of players $S = \{i_1, ..., i_s\}$ in order to minimize the total number of weighted walks in the network, $b(g_{-S},\theta)$. After deleting player i_1, the

[25] Note that we restrict the maximization program to $|S| = s$, given that $d_S(g,\theta)$ is obviously increasing in the size of S.

[26] Note that this sum is independent of the order in which nodes are removed.

resulting number of walks is $b(g,\theta) - d_{i_1}(g,\theta)$. Now, the expression $d_{i_2}(g_{-i_1},\theta)$ counts the number of walks that hit agent i_2 *once agent* i_1 *has been eliminated*, so that we are not counting the same walk twice. Thus, $b(g,\theta) - d_{i_1}(g,\theta) - d_{i_2}(g_{-i_1},\theta)$ is the remaining set of walks after eliminating players i_1 and i_2, keeping in mind that we only want to count each walk once. By the previous argument, also note that the remaining set of weighted walks is the same if we change the order of deletion of these two players, that is:

$$b(g,\theta) - d_{i_1}(g,\theta) - d_{i_2}(g_{-i_1},\theta) = b(g,\theta) - d_{i_2}(g,\theta) - d_{i_1}(g_{-i_2},\theta)$$

Extending this argument to the rest of the players in S, we obtain expression (12).

3.3.2 Example

Consider again the network g in Figure 1 with eleven delinquents and a decay factor $\theta = 0.2$. When $s = 1$, note that Katz-Bonacich *centrality* and our *individual intercentrality* measure are different concepts. The first accounts for the influence of one agent from his position, in terms of the number of agents that he can reach. The second adds the contribution of this agent to the Katz-Bonacich centrality of the others. Hence, individual intercentrality captures the role of each agent as a *broker* in the interactions among the others. For instance, it is easy to check that the key player is 1 because he has the highest individual intercentrality $d_1(g,\theta) = 41.67$. But the player with the highest contribution need not be the one with highest Katz-Bonacich centrality. In particular, individual 2 is more (Katz-Bonacich) central than individual 1: $b_2(g,\theta) = 9.17 > 8.33 = b_1(g,\theta)$.

Consider the case where the required group size is $s = 2$. The next table shows the values of *group intercentrality* $d_S(g,\theta)$ for each possible subset S of size 2 when $\theta = 0.2$. For the sake of simplicity, subsets that yield the same network architecture when they are removed are considered as equivalent:

Removed Group S	$d_S(g,\theta)$
$\{2,7\}^*$	67.22
$\{2,8\}$	64.01
$\{3,8\}$	59.39
$\{1,2\}$	56.66
$\{2,6\}$	50.41
$\{2,3\}$	46.96
$\{3,4\}$	42.15

The key group is $\{2,7\}$, that is, a set of two maximally connected nodes (with five direct contacts each), both connected to the intercentral player 1, and each at a different side of this player. This subset solves the following optimization problem:

$$\max_{i,j} d_{\{i,j\}}(g,\theta) = \max_{i,j} \left(d_i(g,\theta) + d_j(g_{-i},\theta)\right)$$

Suppose that we were to approximate the solution to this optimization problem with a greedy heuristic procedure that *sequentially* picks up the player that maximizes the individual intercentrality at each step. Formally, let

$$i_1^* = \arg\max_{i \in N} d_i(g, \theta)$$

and then, at each step $2 \leq t \leq s$, choose the player i_t^* with maximum intercentrality in the network where the previous players have been deleted, that is,

$$i_t^* \in \arg\max \left\{ d_i(g_{-\{i_1^*,...,i_{t-1}^*\}}, \theta) : i \in N \backslash \left\{ i_1^*, ..., i_{t-1}^* \right\} \right\}$$

breaking possible ties arbitrarily. This greedy algorithm first eliminates player 1, and then any other remaining player (after player 1 has been removed, all the other players have identical positions in the network). Thus, the algorithm returns a group which is not optimal: there are other groups that are better candidates than $\{1, 2\}$. Indeed, in this example, player 1 is not only very intercentral, but also his intercentrality is very much correlated with the intercentrality of others. Hence, being greedy and eliminating it at the first stage reduces the chance of finding highly central players at further stages. And, in fact, player 1 is not part of the key group!

Nevertheless, we have obtained a relatively accurate approximation for the result by a simple greedy algorithm, instead of choosing among all possible pairs of agents. Note that, in this example, the error of this approximation is:

$$\frac{d_{\{2,7\}}(g, \theta) - d_{\{1,2\}}(g, \theta)}{d_{\{2,7\}}(g, \theta)} \approx 16\%$$

In fact, when $s = 2$, this error can be at most 25%. In the next section devoted to algorithmic considerations, we discuss this issue more generally.

3.3.3 Algorithmic considerations

We prove that the key group problem has an inherent complexity that suggests the use of approximation algorithms. In particular, we will study the performance of a greedy procedure where the optimal group is constructed by iteratively choosing an optimal vertex from the network. For a description of $NP-$hard problems and properties, see Garey and Johnson (1978) and Ballester (2004).

Now, we show that the key group problem is NP-hard, even when we want to completely disrupt the game. First, note that if we were to implement a "brute-force" basic algorithm to find a key group of s players, we would have to step over all possible $\binom{n}{s}$ groups of players and compute each particular contribution to the game. This combinatorial procedure may involve up to an exponential number of steps in n. The computational complexity here is mainly combinatorial, that is, while computing the contribution of a given group to the activity of the game is computationally tractable, the fact that this task has to be done an exponential number of times (in the worst case) makes the

problem potentially intractable. $NP-$hardness relates to the difficulty of computationally solving a particular class of problems. Hence, by showing that the key group problem is NP-hard, we show that there is no possible sophisticated algorithm such that, given any network, will return the exact key group in reasonable time.[27] This means that *the key group problem is a NP-hard problem, from the combinatorial perspective*. Nevertheless, we will show below that we can efficiently approximate it.

Proposition 5 *The problem of finding a key group in a network g is NP-hard.*

Since the computational complexity inherent to the key group selection is high, it is suitable to use algorithmic approximations in order to solve real-life problems with large networks.

Consider a *greedy algorithm* that sequentially eliminates in s stages the player with highest intercentrality, that is, let $S^G = \left\{i_1^G, \ldots, i_s^G\right\}$ such that for all $t = 1, \ldots, s$, player i_t^G is the most intercentral in $g_{-S_t}$where $S_t = \left\{i_1^G, \ldots, i_t^G\right\}$. We have the following result:

Proposition 6 *The key group problem can be approximated in polynomial-time by the use of a greedy algorithm, where, at each step t, expression (8) is used to find the agent i_t^G who will become a member of the approximated key group S^G. The error of the approximation can be bounded as follows.*

$$\varepsilon \equiv \frac{d_{S^*}(g,\theta) - d_{S^G}(g,\theta)}{d_{S^*}(g,\theta)} < \frac{1}{e} \approx 36.79\%$$

This proposition shows that the error of approximation of using a greedy algorithm instead of solving directly the key group problem is at most 36.79%.[28] If the approximation error is over 30 percent in most situations then it would be difficult to claim that this result provides a good approximation. Let us now provide some numerical simulations where the bound is calculated for a (large) number of different situations and show that the actual value of the approximation error is in fact rather small. For that, let us consider different scenarios of random networks based on the following variables:

(i) Number of players. We consider two cases: $n = 10$ and $n = 15$. For larger n, combinatorial problems become very important.

(ii) Probability p of a link between any pair of players (i.e. criminals) in the random network. We consider three cases: $p = 0.3$ (sparse networks), $p = 0.5$ (moderate networks) and $p = 0.75$ (dense networks).

[27] This fact is conjectured by nearly all computer scientists who believe that there is no such algorithm for solving any $NP-$hard problem. A simple reason for this is that, after decades of continuous search, no one has found an efficient algorithm for solving any $NP-$hard problem.

[28] As Nemhauser *et al.* (1978) have showed, the error bound obtained is tight. See Appendix A and, in particular, Proposition 10, for definitions and technical details..

(*iii*) Decay factor θ. We consider two cases: networks with small decay factor where long walks matter very little (θ is equal to 10 percent of its upper bound, i.e. $\theta = 0.1/\rho(g)$, where $\rho(g)$ is the spectral radius of the adjacency matrix $\mathbf{G}$) and networks with high decay factor where long walks matter almost as much as short walks (θ is equal to 90 percent of its upper bound, i.e. $\theta = 0.9/\rho(g)$).

(*iv*) Size of the key group k. We consider four cases for $n = 10$ (i.e. $k = 2, 3, 5, 8$) and four cases for $n = 15$ (i.e. $k = 3, 5, 8, 12$).

This means that, in total, we have $2 \times 3 \times 2 \times 4 = 48$ possible scenarios since there are 2 possible n, 3 possible p, 2 possible θ and 4 possible k. In each scenario, we perform a simulation with 100 possible different networks. For each simulation, we first find the exact key group by searching through all possible subsets of players (for instance, for $n = 15$ and $k = 8$, the program searches through all possible $\binom{15}{8} = 6435$ subsets of 8 players) and obtains its intercentrality $d_{S^*}(g, \theta)$; second, we approximate the optimal group using the greedy algorithm and obtains its intercentrality$d_{SG}(g, \theta)$; finally, we calculate the relative error of approximation $\frac{d_{S^*}(g,\theta) - d_{SG}(g,\theta)}{d_{S^*}(g,\theta)}$, which should be below 36 percent according to Proposition 6. The following two tables display the results of our numerical simulations for $n = 10$ and for $n = 15$. The numbers in the tables are the average relative error of approximation (in percentage) over the 100 networks in each scenario.

		$k=2$	$k=3$	$k=5$	$k=8$
$p=0.3$	$\theta = 0.1/\rho(g)$	0.05	0.06	0.12	0
	$\theta = 0.9/\rho(g)$	0.24	0.15	0.13	0
$p=0.5$	$\theta = 0.1/\rho(g)$	0.04	0.06	0.07	0
	$\theta = 0.9/\rho(g)$	0.12	0.08	0.09	0
$p=0.75$	$\theta = 0.1/\rho(g)$	0.01	0.05	0.07	0.02
	$\theta = 0.9/\rho(g)$	0.04	0.05	0.07	0.02

Case 1: $n = 10$

		$k=3$	$k=5$	$k=8$	$k=12$
$p=0.3$	$\theta = 0.1/\rho(g)$	0.08	0.02	0.11	0
	$\theta = 0.9/\rho(g)$	0.11	1.7	0.10	0
$p=0.5$	$\theta = 0.1/\rho(g)$	0.09	0.03	0.06	0.01
	$\theta = 0.9/\rho(g)$	0.07	0.03	0.07	0.02
$p=0.75$	$\theta = 0.1/\rho(g)$	0.01	0.06	0.02	0.02
	$\theta = 0.9/\rho(g)$	0.10	0.06	0.07	0.03

Case 2: $n = 15$

In all cases displayed in the tables, the relative error of approximation is very small, varying between 0 percent (when $k = 8$ for $n = 10$ or $k = 12$ for $n = 15$) and 1.7 percent (for $k = 5$ and $n = 15$). Note that these simulations have been performed for random networks, which tend to be more "symmetric" that networks that are not random. For example, for the bridge network described in Figure 1, the error of approximation was 16 percent. Still, this value is relatively low compared to the upper bound of 36.79 percent. Therefore, we are pretty confident that the use of the greedy procedure can be guaranteed to provide a fairly good approximation S^G for the true solution S^* of the problem.

3.4 Finding the key link

Let us now focus on a different crime policy that targets *links* rather than *individuals*. The aim of this policy is to choose how to remove optimally a link (or a set of links) between two individuals in order to minimize the total delinquency level. In some situations, the limitation of resources or the nature of the problem requires to optimally choose among the set of dependences among players. For instance, a social planner would like to optimally reduce the (communication) externalities among delinquents subject to a restriction in the number r of bilateral influences that can be targeted. This situation can be interpreted as a problem of optimally removing a set of links from the network.

Let us illustrate this policy with real-world examples. As stated above, a link removal means a disruption of the communication between two delinquents. For instance, when a policeman is watching the street, he is somehow disrupting the possible communication between delinquents from the same neighborhood (a link can be understood as communication in a particular place). This policeman is not, however, avoiding communication with other delinquents somewhere else. Another example is to put a delinquent teenager in another school where there are less delinquent.[29] By doing so, this delinquent will stop his activities and communication with other delinquents in the older school. In this section, we will not compare link-removal and player-removal policies since it depends on the costs for the policy maker, and we are not dealing with this issue. The key-link policy should, however, be understood as closely related to the key-player policy since the removal of a player implies the removal of his links *plus* the removal of the isolated player that remains. Removing a set of links is somewhat more flexible because the policy-maker can target links from *different* players.

More formally, for $g' \subset g$, let $l_{g'}(g, \theta)$ be the number of walks in g (weighted by θ) that use some edge in g'. This is the contribution of g' to the total connectivity of g.

Suppose that we need to maximize the change in network activity after removing at most r

[29] See, for example, Ludwig *et al.* (2001) and Kling *et al.* (2005) who study the Moving to Opportunity (MTO) experiment that relocates families from high- to low-poverty neighborhoods. They find that this policy reduces juvenile arrests by 30 to 50 percent of the arrest rate for control groups.

links. Our best choice will consist of r links from the set of all possible links not present in g. Formally, we need to solve:

$$\max_{g' \subseteq g} \{l_{g'}(g, \theta) : |g'| \leq r\}.$$

Consider again the bridge network described in Figure 1. As in the case of the key player, even when $r = 1$, the optimal choice can depend on the strength of complementarities, as shown in the following table:

Removed link $\{i, j\}$	Reduction in $b(g, 0.1)$	Reduction in $b(g, 0.22)$
$\{1, 2\}$	0.59	185.99*
$\{2, 6\}$	0.63*	180.84
$\{2, 3\}$	0.58	164.37
$\{3, 4\}$	0.53	148.95

For moderate values of θ (i.e. $\theta = 0.1$), the key link to be removed is the one between the most central nodes, i.e. delinquents 2 and 6. However, for higher values of θ (i.e. $\theta = 0.22$), intermediate positions become more relevant and the key link is part of the bridge between the two clusters in the network, i.e. delinquents 1 and 2.

Let us now derive more general results. We first deal with the case of *directed* links (non-symmetric networks) since it provides an easier expression of our result. In this case, the planner has more degrees of freedom because it can target specific directed links. Let $h \equiv g \backslash \{ij\}$ be the network g where g_{ij} is set to zero. The following relation holds in this class of networks, for all pair of agents $k, l \in N$:

$$m_{kl}(g, \theta) - m_{kl}(h, \theta) = \theta m_{ki}(h, \theta) m_{jl}(g, \theta) \tag{13}$$

That is, all walks from k to l arrive at i for the first time before crossing ij (so this set of walks occurs in the network h), cross the link ij and then continue from j to l in the network g. Let $l_{ij}(g, \theta)$ be the total contribution of link ij to the centrality of g:

$$l_{ij}(g, \theta) \equiv \sum_{k,l \in N} (m_{kl}(g, \theta) - m_{kl}(h, \theta))$$

Let $\widetilde{b}_i(g, \theta)$ be the Katz-Bonacich *in*-centrality of player i, i.e., the weighted sum of the value of walks entering node i in the network g:

$$\widetilde{b}_i(g, \theta) = \sum_{j=1}^{n} m_{ji}(g, \theta)$$

Lemma 1 *The contribution of a single directed link $ij \in g$ to the total Katz-Bonacich centrality of the network g is given by:*

$$l_{ij}(g, \theta) = \theta \frac{\widetilde{b}_i(g, \theta) b_j(g, \theta)}{1 + \theta m_{ji}(g, \theta)} \tag{14}$$

This expression reflects the asymmetry of players i and j under the assumption of directed links. The effect of a directed link ij depends roughly on the in-centrality of player i and the out-centrality of player j.

When links are *undirected*, the following expression allows us to compute the contribution of a single link $ij \in g$ to the total Katz-Bonacich centrality of the network g. The proof is omitted, being similar to the case of directed links.

Lemma 2 *The contribution of a single undirected link $ij \in g$ to the total Katz-Bonacich centrality of the network g is given by:*

$$\begin{aligned} l_{ij}(g,\theta) &= \theta\left(b_i(h,\theta)b_j(g,\theta)+b_i(g,\theta)b_j(h,\theta)\right) \\ &= \theta\frac{2b_i(g,\theta)b_j(g,\theta)\left(1+\theta m_{ij}(g,\theta)\right)-\theta\left[b_i^2(g,\theta)m_{jj}(g,\theta)+b_j^2(g,\theta)m_{ii}(g,\theta)\right]}{\left[1+\theta m_{ij}(g,\theta)\right]^2-\theta^2 m_{ii}(g,\theta)m_{jj}(g,\theta)} \end{aligned} \quad (15)$$

In order to provide an interpretation, we take a moderate θ. Then, $l_{ij}(g,\theta)$ is proportional to $b_i(g,\theta)b_j(g,\theta)$. This means that, for moderate values of θ, the key link is the one connecting any two nodes with highest Katz-Bonacich centrality. This was the case in the example above where the link $\{2,6\}$ was chosen when $\theta = 0.1$.

Expressions (14) and (15) have obvious advantages, as (8) does in the case of the key player. We can compute the contribution of any link ij from the current data $\mathbf{M}(g,\theta)$ without having to recompute an inverse $\mathbf{M}(g\backslash\{ij\},\theta)$ for each $ij \in g$. These operations are clearly cheaper than the computation of an inverse. This fact becomes critical if we are to approximate the optimal interaction set. The reason is that the function $l_{g'}(g,\theta)$ is submodular[30] in g' so that we can iteratively find the maximum of $l_{ij}(g,\theta)$ using (14) or (15) to obtain quickly a good approximation of the problem:

Proposition 7 *The key interaction set problem can be approximated in polynomial-time by the use of a greedy algorithm where, at each step, expression (14) or (15) is used to find the link ij (with highest $l_{ij}(g,\theta)$) that will become a member of the approximated key interaction set. The error of the approximation can be bounded as follows:*

$$\varepsilon \equiv \frac{l_{g^*}(g,\theta)-l_{g^G}(g,\theta)}{l_{g^*}(g,\theta)} < \frac{1}{e} \approx 36.79\%$$

4 Joining delinquency networks

4.1 Equilibrium networks

In this section, we extend our game in order to allow individuals to choose whether they want to participate in the crime market or not in the first stage. So far, we have assumed that the

[30] See Definition 7 in Appendix A for the definition of a submodular function.

delinquency network was given. In some cases, though, delinquents may have opportunities outside the delinquency network. For instance, petty delinquents may consider entering the labor market and giving up delinquent activities. Here, we expand the model and endogenize the delinquency network by allowing delinquents to take a binary decision on whether to stay in the delinquency network or to drop out of it.[31] Formally, we consider the following two-stage game.

Fix an initial network g connecting agents.

In the first stage, each agent $i = 1, ..., n$ decides to enter the labor market or to become a delinquent. This is a simple binary decision. These decisions are simultaneous. Let $c_i \in \{0, 1\}$ denote i's decision, where $c_i = 1$ (resp. $c_i = 0$) stands for becoming a delinquent (resp. entering the labor market), and denote by $\mathbf{c} = (c_1, ..., c_n)$ the corresponding population binary decision profile. We assume that agents entering the labor market earn a fixed wage (nonnegative scalar) $\omega > 0$. The payoff for delinquents is determined in the second stage of the game.

In the second stage, delinquents decide their effort level, which depends on the first-stage outcome.

Definition 4 *The extended game is a two stage game where:*

- *In stage 1, each player $i \in N$ decides whether to participate ($c_i = 1$) or not ($c_i = 0$) to the crime market.*
- *In stage 2, let S be the set of players who decided to participate. Then, these players play the game in g_S.*
- *The final utilities are:*

$$U_i(S, \mathbf{x}_S, g) = \begin{cases} u_i(\mathbf{x}_S, g_S) & \text{if} \quad i \in S \\ \omega & \text{otherwise} \end{cases}$$

We study the *subgame perfect equilibrium in pure strategies* of this extended game.

Definition 5 *The set S is supported in equilibrium if there exists a ω and a subgame perfect equilibrium where the set of players who decide to participate is S, given the outside option ω. S is also called an (equilibrium) participation pool of the game at the wage level ω.*

Let $\mathcal{E}(\omega)$ be the family of sets supported by ω at equilibrium in the extended game.

The following result characterizes the class of sets that can be supported by some ω.

[31] See Calvó-Armengol and Jackson (2004) for a similar endogenous game of network formation in the context of the labor market, where the binary decision for agents is to enter the labor market network or to drop out.

Proposition 8 *Let $S \subseteq N$ and $\theta\rho(g) < 1$ for all $j \in N\backslash S$. Then, the set S is supported at equilibrium by the outside option ω if and only if:*

$$\max_{j\in N\backslash S} \frac{b_j(g_{S\cup\{j\}},\theta)}{1+b(g_{S\cup\{j\}},\theta)} \leq \frac{1}{1-\pi}\sqrt{\omega\delta} \leq \min_{i\in S} \frac{b_i(g_S,\theta)}{1+b(g_S,\theta)}$$

Remark 2 *Whenever*

$$\omega > \frac{(1-\pi)^2}{4\delta},$$

all agents outside the delinquency pool is an equilibrium, that is, $\varnothing$ is supported as an equilibrium by ω.

Whenever an equilibrium exists, multiplicity of equilibria is a natural outcome of the extensive form game. This multiplicity can arise, for instance, from the symmetric role of some agents in a network.[32]

4.2 Participation game without global substitutability

Suppose that δ is small, that is, we have that the second-stage game is close to a game with strategic complementarities. Let S be a participation pool (not necessarily an equilibrium pool) at some wage ω. In this case, the payoff that an agent $i \in N\backslash S$ obtains by joining S is equal to:

$$x_i^*(g_{S\cup\{i\}}) = \frac{1-\pi}{\delta} b_i(g_{S\cup\{i\}},\theta),$$

that is, it is proportional to its centrality in the network $g_{S\cup\{i\}}$.

Given that the outside option ω is fixed, it is clear that the two-stage game is *supermodular,*[33] in the sense that the payoffs of player i are increasing with respect to participation decisions of other agents. Formally, for all $S \subseteq T \subseteq N$ and $i \in N\backslash T$, it is clear that:

$$b_i(g_{S\cup\{i\}},\theta) \leq b_i(g_{T\cup\{i\}},\theta)$$

because the right-hand side measures a higher number of walks.

This property ensures the existence of equilibrium for any wage ω, as summarized by the following proposition.

[32] Two agents i and j are *symmetric* in a network whenever the network remains with the same structure after exchanging their labels. In this case, if S is supported at equilibrium, $i \in S$ and $j \in N\backslash S$, so is S' where i has been interchanged with j.

[33] A game is supermodular if, for every player i

- his action set S_i is compact,
- his utility $u_i(s_i, s_{-i})$ is upper semi-continous in s_i, s_{-i}.
- his utility has increasing differences in (s_i, s_{-i}).

For a broad description of supermodular games and their applications, see Amir (2005), Topkis (1998) and Vives (2005).

Proposition 9 *When δ is small, the extended game has at least one equilibrium participation pool.*

One may be interested in providing all the possible equilibria of the game when supermodularity holds. Echenique (2007) provides a useful tool to list all the equilibria of a game with complementarities.

The intuitive idea here is that substitutability is low enough to allow for increasing differences in utility of agents in their decisions to enter the participation pool.

4.3 Finding the key player with criminal participation decision

Given that this game usually displays multiple subgame perfect equilibria in the endogenous delinquency network game, we define $x^*(g, \omega)$ to be the maximum aggregate equilibrium delinquency level when the delinquency network is g and the labor market wage is ω. This delinquency level is equal to the total amount of delinquency in the worst case scenario of maximum delinquency.

Consider some binary decision profile $\mathbf{c}$. Let i be an active delinquent, that is $c_i = 1$. Suppose that delinquent i switches his current decision to $c_i = 0$, that is, delinquent i drops out from the delinquency pool and enters the labor market instead. The binary decision profile then becomes $\mathbf{c} - \boldsymbol{\nu}^i$, and the new set of active delinquents is $C(\mathbf{c} - \boldsymbol{\nu}^i) = C(\mathbf{c}) \backslash \{i\}$. The drop out of delinquent i from the delinquency pool also alters the network structure connecting active delinquents, as any existing direct link between i and any other delinquent in $C(\mathbf{c})$ is removed. The new network connecting active delinquents is then $g(\mathbf{c})^{-i} = g(\mathbf{c} - \boldsymbol{\nu}^i)$, and the aggregate delinquency level becomes:

$$x^*(\mathbf{c} - \boldsymbol{\nu}^i) = \frac{1 - \pi}{\delta} \frac{b\left(g(\mathbf{c} - \boldsymbol{\nu}^i), \theta\right)}{1 + b\left(g(\mathbf{c} - \boldsymbol{\nu}^i), \theta\right)}$$

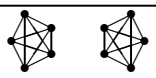
The key player problem acquires a different shape in the setting with endogenous formation of delinquency pools. Initially, the planner must choose a player to remove from the network. Then, players play the two-stage delinquency game. First, they decide whether to enter the delinquency pool or not. Second, delinquents choose how much effort to exert. In this context, there is an added difficulty to the planner's decision. The removal of a player from the network affects the rest of the players' decisions to become active delinquents. This fact should be taken into account by the planner in order to attain an equilibrium with minimum total delinquency. The right choice of the key player should be based upon the remaining delinquency pool that will result from that decision, that is, what the remaining players will decide concerning their delinquent activities.

We show, with the help of an example, that there is no trivial geometric recipe for the key player problem in this case.

Consider again the network in Figure 1 with eleven players. Recall that, when $\theta = 0.2$ and the network of delinquents is exogenously fixed (or, equivalently, the outside option is $\omega = 0$), the key player was the player acting as a bridge, i.e. delinquent 1. If we now consider the endogenous delinquency network formation in the two-stage game, the results may differ. Indeed, for low wages,

player 1 is also the key player and the resulting equilibrium network is the whole remaining network, that is, ten delinquents remain and are split into two fully connected cliques of five delinquents. However, when ω becomes higher, delinquent 2 becomes the key player[34] and the equilibrium network now encompasses six different players. It consists of a clique of five fully intraconnected players together with player 1.

These results are summarized in the following table, which gives, for two different values of ω, the key player, the highest aggregate delinquency that results from eliminating this key player, and the equilibrium delinquency network.

	$\omega = 0.001$	$\omega = 0.003$
$x^*(g_{-1}, \omega)$	0.7843	0.7843
$x^*(g_{-2}, \omega)$	0.7847	0.7785
Key Player	1	2
Final delinquency pool		

Intuitively, when outside opportunities are high enough, all players from the same side of the player being removed do not have enough incentives to enter the delinquency pool at the first stage of the game. Hence, we do not get a "large" equilibrium with many players, and this constitutes an advantage for the planner who will choose to delete node 2. This example implicitly explains how one policy (providing a higher ω) increases the effectiveness of another policy (choosing the key player) in order to reduce delinquency. These policies are *complementary* from the point of view of their effects on total delinquency, although we are aware that they may be substitute if we had considered a budget-restricted planner who had to implement costly policies.

4.4 The key-group problem with criminal participation decision

In the simple case without outside option, the choice of the key group was based on the contribution of that group to the connectivity (total Katz-Bonacich centrality) of the network. In the context of games with criminal participation decision, an additional criterion should be taken into account: the fact that a removal of some players may induce further voluntary moves of other players in the network. Thus, the choice of the optimal target can change accordingly, and differ from the usual key group prescription when all players participate. We analyze the interplay between the optimal target and an outside option that acts as a participation threshold in the new game.

The issue of existence will be relaxed in this section by assuming that we are dealing with a wage ω such that, for any subgame in the subnetwork g_T, with $T \subseteq N$, there exists an equilibrium participation pool S supported by ω. On the other hand, multiplicity of equilibria makes it difficult to adopt a particular approach in order to assess the efficacy of a particular policy. In this paper, we

[34] In fact, any player except player 1 is the key player for $\omega = 0.003$.

focus on a extreme approach where the removal of a set of players from the network is evaluated by comparing the maximum equilibrium of the original game and the resulting game, that is, outcomes with maximum total activity x^*.

Definition 6 *Given an extended game with wage ω and $T \subseteq N$, the remaining family after eliminating S is defined as:*

$$P(\omega, S) = \{T \subseteq N \backslash S,\, T \in \mathcal{E}(\omega)\}$$

In words, a set $T \subseteq N\backslash S$ is in the remaining family after eliminating S whenever T is a participating pool in the restricted extended game played in g_{-S}. This definition is just capturing the the posterior behavior of players after S's removal.

For a candidate set S to be eliminated, let $P_m(\omega, S) \in \arg\max_{T \in P(,\omega,S)}\{b(g_T, \theta)\}$ be a *maximum equilibrium* participation pool when the set S is eliminated. It is a pool where the maximum activity is achieved. Then, the choice of a key group S^* of size s is:

$$S^* \in \arg\min_{|S| \leq s} b(g_{P_m(\omega,S)}, \theta)$$

Let us illustrate this with the network described in Figure 1. We study the problem of eliminating one player ($s = 1$) when $\theta = 0.2$. When we analyze the extended game with criminal participation decision, it is crucial for the planner to consider the possible transitions between different pools of delinquents. In particular, there are now three effects that should be taken into account when choosing the set of players to be removed:

(i) A *direct effect* due to the reduction of their initial delinquent activity. The choice is here biased towards the most Katz-Bonacich central players.

(ii) An *indirect effect* due to the (lower) incentives of the remaining players. In this dimension, group-intercentrality is the relevant variable to consider.

(iii) A possible *snow-ball effect* because the removal of a player may induce a process where the remaining players (sequentially) find it profitable to leave the pool of delinquents and to participate to the labor market. This effect depends on the magnitude of the outside option ω.

The next table summarizes the sets S that are sustainable in equilibrium for the games played in g_{-1} (i.e. when delinquent 1 is removed) and g_{-2} (i.e. when delinquent 2 is removed), specifying the range $[\omega_L, \omega_H]$ of wages that support for each S an equilibrium participating pool (we can have

multiplicity of equilibria). We just specify distinct (up to network isomorphism) equilibrium pools:

	g_{-1}			g_{-2}		
Pool S	ω_L	ω_H	$b(g_S, \theta)$	ω_L	ω_H	$b(g_S, \theta)$
$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$	0.00	12.50	50.00	–	–	–
$\{2, 3, 4, 5, 6\}$	0.50	12.50	25.00	–	–	–
$\{1, 6, 7, 8, 9, 10, 11\}$	–	–	–	1.02	1.70	39.47
$\{1, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$	–	–	–	0.00	4.17	51.33
$\{1, 7, 8, 9, 10, 11\}$	–	–	–	1.70	7.03	36.25
$\{7, 8, 9, 10, 11\}$	–	–	–	7.03	12.50	25.00
$\{1, 3, 4, 5, 6\}$	–	–	–	0.95	1.25	12.37
$\{3, 4, 5, 6\}$	–	–	–	1.25	3.12	10.00

To illustrate the results given in this table, let us consider the case when $\omega = 5$. If delinquent 1 is removed then the highest equilibrium pool consists of all delinquents but 1, that is $N\backslash\{1\} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. This is exactly the same choice as in the case without criminal participation decision, which is based on the intercentrality index. If, on the contrary, delinquent 2 is removed then for the wage $\omega = 5$, it is easily seen that the maximum equilibrium pool becomes $\{1, 7, 8, 9, 10, 11\}$. In other words, with the exception of 1, all the delinquents directly connected to 2 find it not profitable to become delinquents and instead prefer to participate to the labor market. As a result, when the wage is $\omega = 5$ and there is no criminal participation decision, then the key player is delinquent 2 because its deletion from the network has the highest impact on the incentives of other players to become delinquent. In other words, by deleting delinquent 2 instead of 1, fewer individuals will become delinquent. This will also lead to a higher decrease in the aggregate level of crime.

If we now consider a much lower wage, say $\omega = 0.4$, then removing delinquent 1 or 2 will have the same effect on individuals' participation in criminal activities. Indeed, in both cases, all individuals will find it profitable to be delinquent since when 1 is removed, $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ will be criminals in equilibrium while, when 2 is removed, the equilibrium pool of delinquents is: $\{1, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. This suggests that the effectiveness of a key-player policy should not only be measured by the direct and indirect effects on delinquent activities but also by the group interactions it engenders in terms of participation to the labor market.

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Appendix A. Notation and definitions.

A.1. Matrix and vector notations

Matrices and vectors will be denoted in bold letters, like $\mathbf{A}$ and $\mathbf{x}$, respectively. If not explicitly stated, all matrices are square. The *entries* of $\mathbf{A}$ (matrix) and $\mathbf{x}$ (vector) are written as a_{ij} and x_i, respectively.

The *transpose* of $\mathbf{A}$ and $\mathbf{x}$ are $\mathbf{A}^T$ and $\mathbf{x}^T$. The matrix $\mathbf{A}^k$ is the k-th *power* of $\mathbf{A}$, and its (i,j)-entry is written $a_{ij}^{[k]}$.

The identity matrix is $\mathbf{I}$. The symbol $\mathbf{0}$ will be used for the *zero* vector. The symbol $\mathbf{1}$ will be used for the *one* vector, where every entry is 1. Given a vector $\mathbf{x}$, the scalar $x \equiv \mathbf{1}^T \cdot \mathbf{x}$ is the sum of all its entries, and $x_S \equiv \mathbf{1}^T \cdot \mathbf{x}_S$, where $\mathbf{x}_S$ is the restriction of the vector $\mathbf{x}$ to the indices in S.

An *eigenvalue* of a matrix $\mathbf{A}$ is a complex number μ satisfying $\mathbf{A} \cdot \mathbf{v} = \mu \mathbf{v}$ for some complex vector $\mathbf{v}$. Let $\mathcal{S}(\mathbf{A})$ (called the *spectrum* of the matrix $\mathbf{A}$) be the set of all eigenvalues of $\mathbf{A}$.

A.2. Networks

A *network* (*graph*) g consists of a set of *agents* (*vertices* or *nodes*) N and a set of weighted *links* (*edges*) between them, where $g_{ij} \geq 0$ is the weight assigned to the link ij. We may represent a network by means of a nonnegative square *adjacency matrix* $\mathbf{G} = (g_{ij})_{i,j \in N}$. Without loss of generality, we will consider networks where $g_{ij} \in [0,1]$. A network g_{ij} is *un-weighted* when $g_{ij} \in \{0,1\}$, for all $i,j \in N$.

The network g is *symmetric* (or undirected) when its adjacency matrix $\mathbf{G}$ is *symmetric*, that is, $g_{ij} = g_{ji}$ for all $i,j \in N$.

We refer to the agents i and j as being *directly linked* in the network g, whenever $g_{ij} > 0$.

A link ij is *incident* with the vertex $v \in N$ in the network g whenever $i = v$ or $j = v$.

A *walk* in g of length k from i to j is a sequence $p = \langle i_0, i_1, ..., i_k \rangle$ of agents such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and i_p and i_{p+1} are directly linked, for all $0 \leq p \leq k-1$. Agents i and j are said to be *indirectly linked* in g if there exists a walk from i to j in g. An agent $i \in N$ is *isolated* in g if $g_{ij} = 0$ for all j. The network g is said to be *empty* when all its agents are isolated.

We say that a walk p *crosses* or *hits* agent i if i is in the sequence defined by the walk. The walk p *covers* the set $S \subseteq N$ if p crosses every agent $i \in S$.

We say that network g' *is a (proper) subnetwork of* g, written $g' \subseteq g$ ($g' \subset g$), whenever $N' \subseteq N$ and $\mathbf{G}' \leq \mathbf{G}_{N'}$ ($\mathbf{G}' \lneq \mathbf{G}_{N'}$).

Given a network g and a set $S \subseteq N$, we say that g_S is the *subnetwork of* g *induced by* S whenever the adjacency matrix of g_S is $\mathbf{G}_S$. We write g_{-S} to denote the network $g_{N \setminus S}$, that is g_{-S} is the network that results after eliminating all the agents in S.

The *spectral radius* of a network g is defined as:

$$\rho(g) = \max_{\mu \in \mathcal{S}(\mathbf{G})} |\mu|$$

where $|\mu|$ is the modulus of the (complex) eigenvalue μ of the matrix $\mathbf{G}$. When g is undirected, all the eigenvalues of $\mathbf{G}$ are real and $\rho(g)$ is called the *index* of the network g.

We adapt some results from spectral graph theory[35] and algebra into our framework.

Lemma 3 *The following properties hold for any network g:*

1. *If $g' \subseteq g$, then $\rho(g') \leq \rho(g)$.*
2. *$\rho(g_S) \leq \rho(g)$ for all $S \subseteq N$.*

A.3. Maximization of submodular functions

The optimal choice of the group of players requires, at least potentially, the study of all possible combinations of subsets of N. Thus, a computational approach is required. Let $z : 2^N \to \mathbb{R}$ be a set function. Consider the problem of solving (7), that is,

$$\max_{S \subseteq N} \{z(S) : |S| \leq s\}. \tag{16}$$

Definition 7 *The set function $z : 2^N \to \mathbb{R}$ is submodular (supermodular) if for all $S, T \subseteq N$,*

$$z(S) + z(T) \underset{(\leq)}{\geq} z(S \cup T) + z(S \cap T)$$

Without loss of generality we can normalize z such that $z(\varnothing) = 0$. We only consider nondecreasing functions:

$$z(S) \leq z(T) \text{ for all } S \subseteq T \subseteq N$$

although the following results can be adapted to non-monotonic functions. Let us denote individual contributions by:

$$\rho_i^z(S) = z(S \cup \{i\}) - z(S)$$

In fact, the set function z is submodular if individual contributions are increasing with respect to set containment.

Remark 3 *The set function $z : 2^N \to \mathbb{R}$ is submodular if and only if for all $S \subseteq T \subseteq N$ and $i \in N \backslash T$:*

$$\rho_i^z(S) \underset{(\leq)}{\geq} \rho_i^z(T)$$

[35] Cvetković et al. (1997) is the main reference for spectral graph theory.

The problem of maximizing a submodular function, or equivalently, minimizing a supermodular function, is NP-hard, in general. Nemhauser *et al.* (1978) propose a polynomial-time greedy heuristic for approximating this kind of problem. At each step, the algorithm augments the solution set with the agent with the highest contribution:

Let $S_0 = \varnothing$. At step t set $S_t = S_{t-1} \cup i_t$, where $i_t \in \arg\max_{i \in N \setminus S_{t-1}} \rho_i^z(S_{t-1})$. Stop whenever $\rho_{i_t}^z(S_{t-1}) \leq 0$ or $|S_t| = s$.

We summarize part of their results in the following proposition. Let Z be the optimal value of (16) and Z^G be the value obtained by applying the greedy algorithm.

The following results allow us to construct the proof of Proposition 6.

Proposition 10 *If the greedy heuristic is applied to the problem (16), where z is submodular, then the approximation error is bounded like:*

$$\varepsilon \equiv \frac{Z - Z^G}{Z} \leq \left(\frac{s-1}{s}\right)^s < \frac{1}{e} \approx 36.79\% \tag{17}$$

Lemma 4 *The function $d_S(g, \theta)$ is submodular in S.*

Proof. Take $S \subseteq T \subseteq N$. Let $b_{ji}^{[k]}(g)$ denote the number of k-walks starting at j and crossing i in the network g. Then, for all $i \in N \setminus T$:

$$\begin{aligned}
d_{S\cup\{i\}}(g,\theta) - d_S(g,\theta) &= \big(b(g,\theta) - b(g_{-(S\cup\{i\})},\theta)\big) - (b(g,\theta) - b(g_{-S},\theta)) \\
&= b(g_{-S},\theta) - b(g_{-(S\cup\{i\})},\theta) \\
&= d_i(g_{-S},\theta) \\
&= \sum_{k=0}^{\infty} \theta^k \sum_{j \in N\setminus S} b_{ji}^{[k]}(g_{-S}) \\
&\geq \sum_{k=0}^{\infty} \theta^k \sum_{j \in N\setminus T} b_{ji}^{[k]}(g_{-T}) \\
&= d_i(g_{-T},\theta) \\
&= d_{T\cup\{i\}}(g,\theta) - d_T(g,\theta).
\end{aligned}$$

∎

Appendix B. Proofs

Proof of Proposition 1. We would like to apply Theorem 1 of Ballester *et al.* (2006). First, observe that the utility function $u_i(x_1, ..., x_n)$ in Ballester *et al.* (2006), defined by their equation (3), can be written as:

$$\begin{aligned} u_i(x_1, ..., x_n) &= \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j \\ &= \alpha x_i - \frac{1}{2}(\beta - \gamma) x_i^2 - \gamma x_i^2 - \gamma \sum_{j \neq i}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j \\ &= \alpha x_i - \frac{1}{2}(\beta + \gamma) x_i^2 - \gamma \sum_{j \neq i}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j \end{aligned} \tag{18}$$

Second, our utility function $u_i(x,g)$ defined by (3) is equivalent to the utility function $u_i(x_1, ..., x_n)$ in Ballester *et al.* (2006), now defined by (18), if and only if:

$$\alpha = 1 - \pi \quad , \quad \frac{1}{2}(\beta + \gamma) = \delta \quad , \quad \gamma = \delta \quad , \quad \lambda = \pi\phi$$

which is equivalent to

$$\alpha = 1 - \pi \quad , \quad \beta = \gamma = \delta \quad , \quad \lambda = \pi\phi$$

Now since by (4), $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$, for all $i \neq j$ with $\underline{\sigma} \leq 0$, then we can use Corollary 1 in Ballester *et al.* (2006), and the condition on eigenvalue: $\beta > \lambda\sqrt{g + n - 1}$ can now be written as: $\pi\phi\sqrt{g + n - 1} < \delta$. ∎

Proof of Proposition 2. It suffices to apply Theorem 2 of Ballester *et al.* (2006) to our framework. ∎

Proof of Proposition 3. It suffices to apply Theorem 3 of Ballester *et al.* (2006) to our framework. ∎

Proof of Proposition 4. Simple algebra leads to:

$$\eta_i(g, \theta) = \frac{\frac{d_i(g,\theta)}{1 + b(g,\theta) - d_i(g,\theta)}}{\frac{1}{n}\sum_{j=1}^{n} \frac{d_j(g,\theta)}{1 + b(g,\theta) - d_j(g,\theta)}}, \text{ for all } i = 1, ..., n.$$

By definition, $d_{i^*}(g, \theta) \geq d_i(g, \theta)$, for all $i = 1, ..., n$. This implies that:

$$\frac{1 + b(g, \theta) - d_{i^*}(g, \theta)}{1 + b(g, \theta) - d_j(g, \theta)} \leq 1, \text{ for all } j = 1, ..., n,$$

and, thus $\eta_{i^*}(g, \theta) \geq d_{i^*}(g, \theta)/\overline{d}(g, \theta)$. Noting that $d_{i^*}(g, \theta) \geq \overline{d}(g, \theta) + \sigma_{\mathbf{d}}(g, \theta)$, we can conclude. ∎

Proof of Proposition 5.

A *vertex cover* of a network g is a subset of vertices $S \subseteq V(g)$ such that every link $ij \in g$ is incident with some vertex in S. A *minimum* vertex cover is vertex cover of minimum size. Note that in any minimum vertex cover, any removal of a vertex from the set will make it fail to cover all the links. The problem of finding a *minimum* vertex cover in a network is known to be NP-hard (Karp, 1972). We show that we can solve this difficult problem by transforming it, in at most n-steps, into a key-group problem, concluding that finding a key-group is also hard.

Lemma 5 *The set S^* is a vertex cover of g if and only if $S^*$is a key group of size $|S^*|$ that disrupts the network g in the game.*

Proof. Let S^* be a vertex cover of g. Obviously, if we are asked to remove $|S^*|$ players in order to minimize activity, the set S^* would be a solution (a key group) that leaves all nodes isolated in the network. Conversely, if S^* is a key group that leaves all nodes isolated, then it must be because it is a vertex cover (all links are incident to nodes in S^*).

This means that if we want to find a minimum vertex cover of a network, we have to iteratively find the key groups of sizes 1,2,... until the network is completely disrupted at some stage $k \leq n$ (the number of iterations is at most n). In this stage k, we have found the key group S_k^*, which is a minimum vertex cover. ∎

Proof of Proposition 6. The result is a direct consequence of Proposition 10 and Lemma 4 in Appendix A.

Proof of Lemma 1. We can specialize (13) to compute $m_{ki}(h,\theta)$ as:

$$\begin{aligned} m_{ki}(g,\theta) - m_{ki}(h,\theta) &= \theta m_{ki}(h,\theta) m_{ji}(g,\theta) \\ m_{ki}(h,\theta) &= \frac{m_{ki}(g,\theta)}{1+\theta m_{ji}(g,\theta)}, \end{aligned}$$

and, substituting it back to (13),

$$m_{kl}(g,\theta) - m_{kl}(h,\theta) = \theta \frac{m_{ki}(g,\theta) m_{jl}(g,\theta)}{1+a m_{ji}(g,\theta)} \text{ for all } k,l \in N.$$

Summing over all k and l, the result follows. ∎

Proof of Proposition 8. The conditions $\theta\rho(g) < 1$ implies that $b_j(g_{S\cup\{j\}},\theta)$ is well-defined for all $S \subset N$ and $j \in N\backslash S$. Given that $\rho(g_{S\cup\{j\}}) \geq \rho(g_S)$, $b_i(g_S,\theta)$ is all also well-defined for all $i \in S$. On the other hand, by Proposition 1, this also implies the uniqueness of the Nash equilibrium in the second stage game defined, respectively, by g_S and $g_{S\cup\{j\}}$, for all $j \in N\backslash S$:

$$x_i^*(g_S) = \frac{1-\pi}{\delta\left(1+b(g_S,\theta)\right)} b_i(g_S,\theta) \text{ for all } i \in S \tag{19}$$

$$x_j^*(g_{S\cup\{j\}}) = \frac{1-\pi}{\delta\left(1+b(g_{S\cup\{j\}},\theta)\right)} b_i(g_{S\cup\{j\}},\theta) \text{ for all } i \in S \tag{20}$$

Now, uniqueness in the second-stage allows us to concentrate on the pure strategy Nash equilibria of the whole game where no agent j outside a sustainable S would be willing to enter the game in the network g_S to obtain $u_j\left(x^*(g_{S\cup\{j\}}), g_{S\cup\{j\}}\right)$; and no agent $i \in S$ would be better off by obtaining ω, rather that $u_i\left(x^*(g_S), g_S\right)$. Formally, a set S is supported by ω at equilibrium if and only if:

$$\max_{j\in N\setminus S} u_j\left(x^*(g_{S\cup\{j\}}), g_{S\cup\{j\}}\right) \leq \omega \leq \min_{i\in S} u_i\left(x^*(g_S), g_S\right).$$

The result follows by using (19) and (20), and applying simple algebra to compute the utilities. ∎

Proof of Proposition 9. We provide an instance of participation pool, by construction. Starting with an empty pool $S_0 = \varnothing$, at each step t, set $S_t = S_{t-1} \cup i_t$, where i_t is any player such that:

$$\sqrt{\frac{2\omega}{\beta}} \leq \frac{\alpha}{\beta} b_{i_t}(g_{S_t}, \theta)$$

This means that, at each step, players that want to enter the pool do so. Stop whenever there is no such agent i_t. It is clear that S_t (probably empty) is an equilibrium pool. The main implication of supermodularity is that this sequential decisions cannot be rolled-back: if an agent decided to enter the pool, then it must be profitable for him to stay after more agents have decided to participate. ∎